Nonlinear Waves

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Mathematics & Statistics, UNC Wilmington

Graduate Seminar

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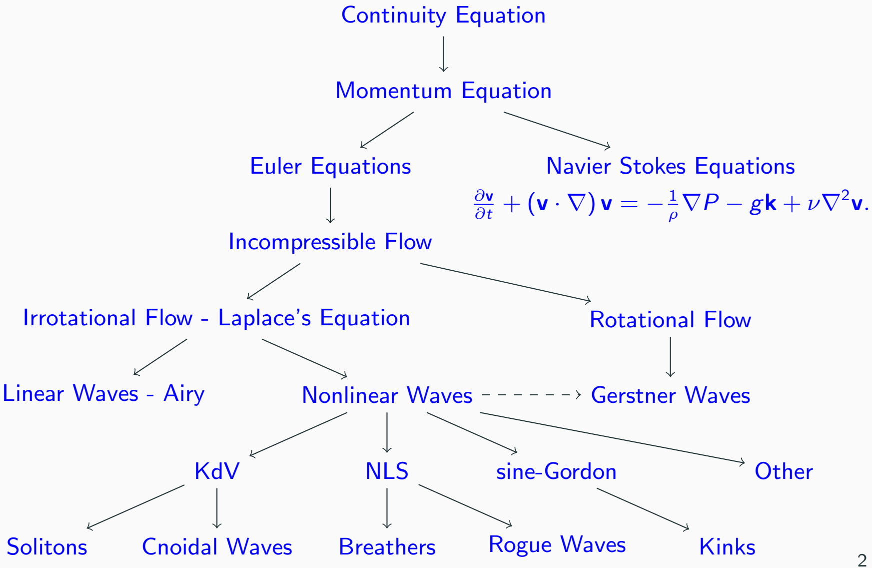
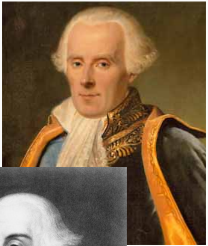
The Great Wave off Kanagawa by Hokusai

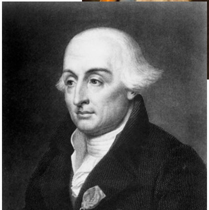
Linear and Nonliner Waves



Water Waves - Before 1800

- 1687 Isaac Newton - 1st attempt at theory of waves.
- 1738 D. Bernoulli, fluid flow in pipes.
- 1744 d'Alembert published a companion volume to his first work, the *Traité de l'équilibre et du mouvement des fluides.*
- 1757 Euler published 3 papers on hydrodynamics, E225: *Principes généraux de l'état d'équilibre des fluides.*, E226: *Principes généraux du mouvement des fluides*, E227: *Continuation des recherches sur la théorie du mouvement des fluides.*
- 1776 Laplace, mostly Theory of Tides.
- 1781 Lagrange (1786, 1788)
  Mémoire sur la Théorie du mouvement des fluides
  - linearized equations for small amplitude waves
  - long waves in shallow water ($\lambda \gg h$).
  - $v = \sqrt{gh}$, independent of wavelength.
  - Used Lagrangian vs Eulerian coordinates.





Euler Equations

- Continuity Equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0.$$

- Momentum Equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} = -\frac{1}{\rho} \nabla P - g\mathbf{k}.$$

- For incompressible flow, ρ is constant giving

$$\nabla \cdot \mathbf{v} = 0.$$

- Irrotational flow,

$$\nabla \times \mathbf{v} = 0.$$

Leonhard Euler (1707-1783)

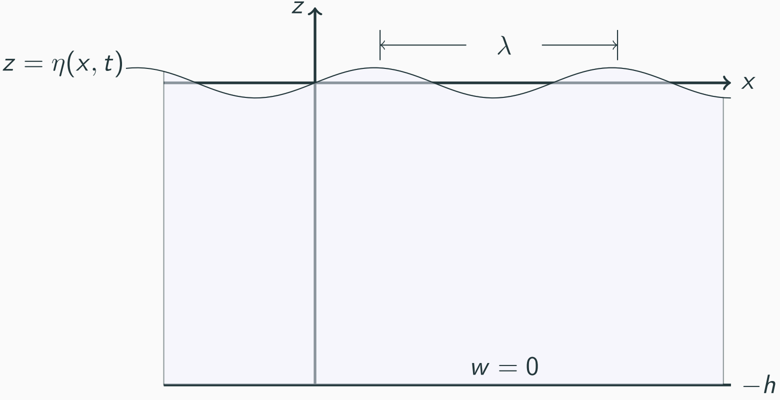
- Potential function,

$$\mathbf{v} = \nabla \phi(\mathbf{x}, t).$$

- Laplace's Equation

$$\nabla^2 \phi = 0.$$

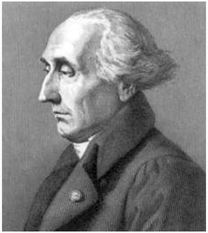
Surface Waves



Plane surface wave with a flat bottom.

Lagrange, Laplace and Water Waves

- Pierre-Simon Laplace (1749-1827) made progress but was disregarded, 1776.
- Lagrange independently derived linearized equations for small amplitude waves, 1781, 1786, 1788.
- Determined wave speed, $v - \sqrt{gh}$, h is liquid depth.
- Wave speed independent of wavelength if $\lambda \gg h$. - Shallow water approximation.
- Laplace posed the initial value problem. Later taken up by Cauchy and Poisson.
- Small displacements governed by "Laplace's equation."



Joseph-Louis Lagrange (1736-1813)

Gerstner Waves - 1802

Gerstner's solution is given by

$$\begin{aligned} X &= a + \frac{1}{k} e^{kb} \sin k(a+ct), \\ Y &= b - \frac{1}{k} e^{kb} \cos k(a+ct) (1) \end{aligned}$$

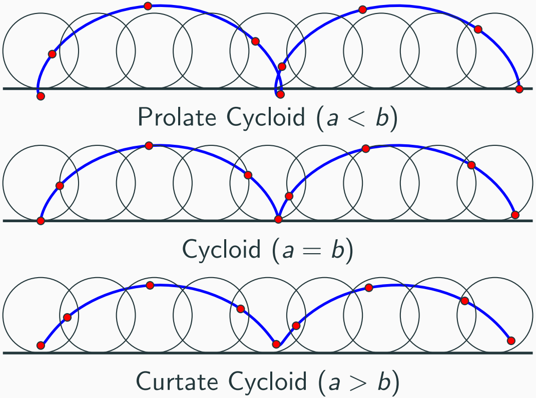
Here, k is related to the wavelength and c is the wave speed.
The surfaces of elevation are trochoids and the limiting cusped case is the free surface and it tends to a cycloid.
The three types of trochoids are parametrically given by

$$x = a\phi - b\sin\phi, \quad y = a - b\cos\phi.$$

František Josef Gerstner (1756-1832)

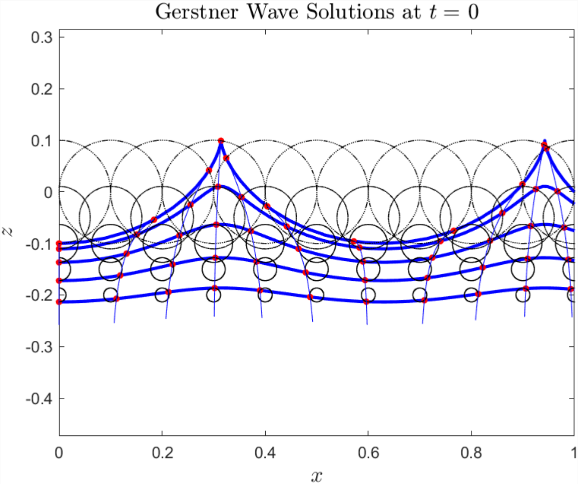
As a wheel of radius a rolls without slipping [$\phi = \omega t$, $v = a\omega$], the curves are followed by a point at a distance b from the center of the wheel.

Plots of prolate cycloid, cycloid, and curtate cycloid



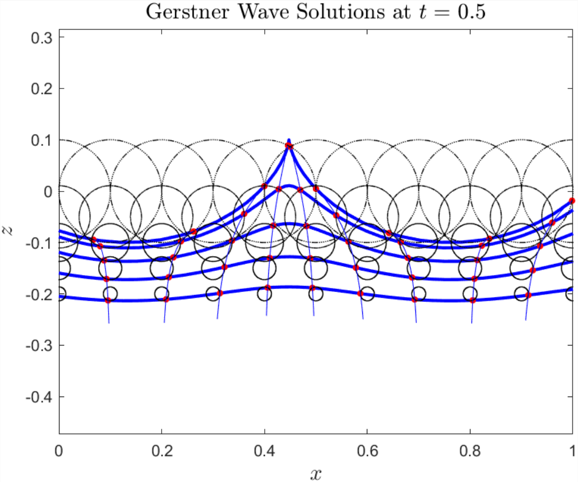
Plots of the points for fixed $m = 10$, $b \leq 0$ at $t = 0$. It consists of horizontal constant b curves and vertical constant a curves. The particle paths consist of circles with centers at (a, b) and radii $\frac{1}{k}e^{kb}$ for $b < 0$.

Gerstner Waves Cross Section



Plots of the points $\left[a + \frac{e^{mb}}{m} \sin m \left(a + \sqrt{\frac{g}{m}} t\right), b - \frac{e^{mb}}{m} \cos m \left(a + \sqrt{\frac{g}{m}} t\right)\right]$ at different times for fixed $m = 10$, $b \leq 0$. The grid consists of horizontal constant b curves and vertical constant a curves.

Gerstner Waves Cross Section



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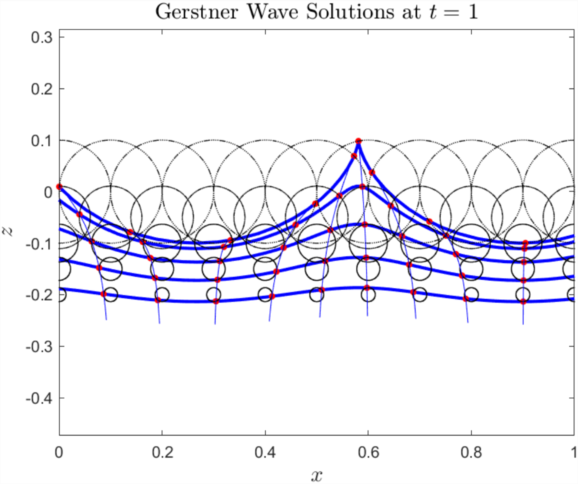

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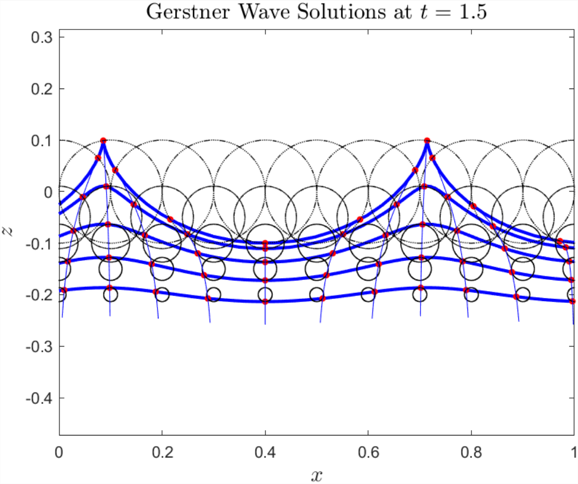
Gerstner Waves Cross Section



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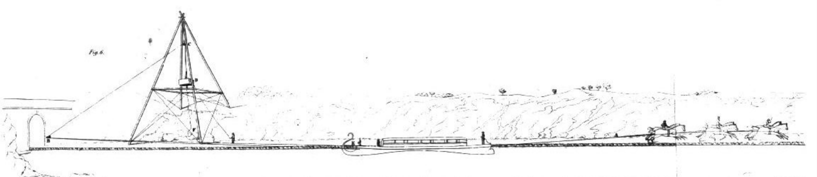
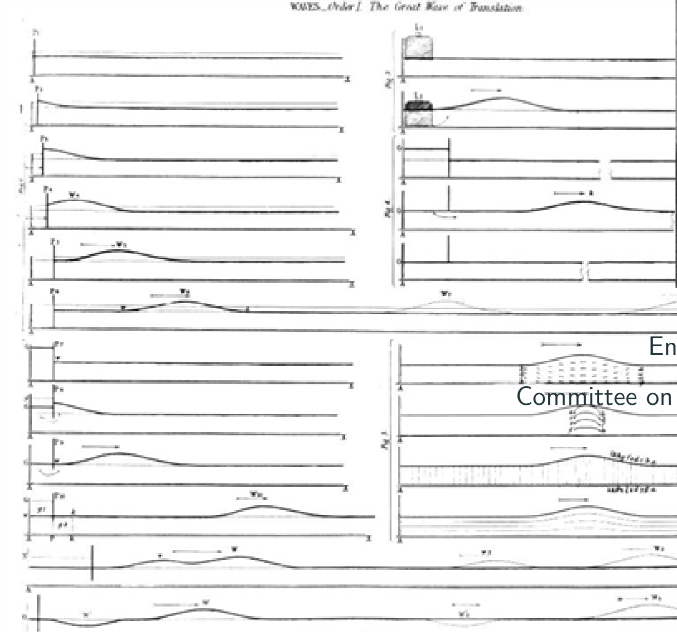
The Great Wave of Translation - 1834

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour [14 km/h], preserving its original figure some thirty feet [9 m] long and a foot to a foot and a half [300-450 mm] in height. Its height gradually diminished, and after a chase of one or two miles [2-3 km] I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation." - John Scott Russell





Engineer, Edinburgh

Committee on Waves, 1837, 1844

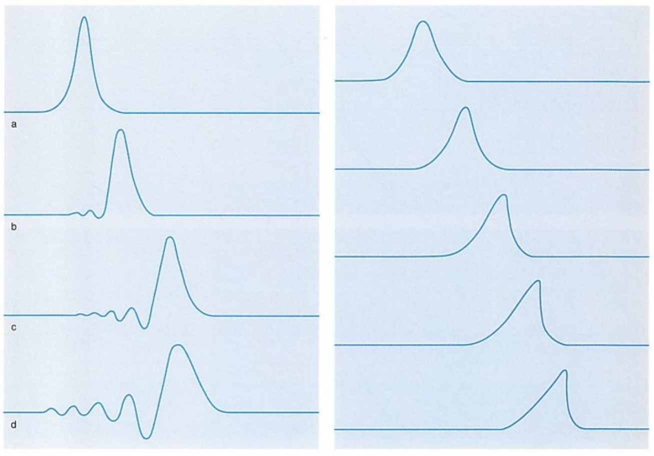
Used 30 ft tank

$v^2 = g(h + a)$

Re-enactment 7/12/95 - Union Canal

Named Scott Russell Aqueduct

Dispersion vs Nonlinearity



Dispersion causes waves to spread and flatten while nonlinearity cause waves to be more compact and steepen.

G. B. Airy and G. G. Stokes

Green, Kelland, Airy and Earnshaw published on water waves, 1840s.
George Green - 1838, "On the motion of waves in a variable canal of small depth and width," 1839, "Note on motion of waves in canals," WKB analysis.

George Biddle Airy (1801-1892)

- 1826 Lucasian Chair, 2 yrs.
- 1835-1881, appointed Astronomer Royal.
- *Tides and Waves,* 1841.
- Linear Water Waves.

Sir George Gabriel Stokes (1819-1903)

- Undergrad, Cambridge 1837.
- Coached by William Hopkins.
- 1841 Senior Wrangler, Smith Prize.
- 1849 Lucasian Chair, 54 yrs.
- Stokes' Theorem, Smith Exam, 1854.

Navier-Stokes Equation

Add friction - $\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P - g\mathbf{k} + \nu\nabla^2\mathbf{v}.$

- 1821 *Sur les lois des mouvements des fluides, en ayant égard à l'adhésion des molecules*
- 1823 *Sur Les Lois du Mouvement des Fluides* (in print, 1827),
- Both read at L'Académie on March 18th, 1822.

- Augustin Cauchy, Siméon Poisson, and Adhémar Barré de Saint-Venant provided other approaches.
- G. G. Stokes, 1845, "On the theories of the internal friction of fluids in motion."
- Viscosity of the air flowing around the pendulum.- Very practical!

Claude-Louis Navier (1785-1836)

Stokes Waves

In 1847, Stokes used perturbation series to obtain approximate periodic solutions to nonlinear wave motion. [See his paper here. Note that Stokes describes J. S. Russell's solitary waves as one motivation in this study.] He considered waves of large enough amplitude that nonlinearity could be taken into account. He did so by expanding solutions to third order in $\epsilon = ka$, the product of the amplitude and wave number. Stokes obtained

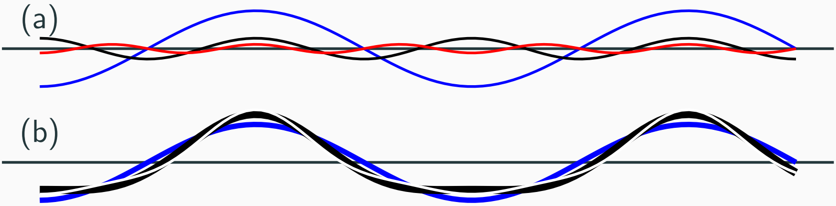
$$z = a\left(\cos k(x-ct) + \frac{1}{2}\epsilon\cos 2k(x-ct) + \frac{3}{8}\epsilon^3\cos 3k(x-ct)\right), \quad (2)$$

where

$$c = \left(1 + \frac{1}{2}\epsilon^2\right)\sqrt{\frac{g}{k}}.$$

We can plot these terms and see the effects on the shape of the surface wave.

Stokes' Wave Corrections



Plots of the Stokes' wave corrections in Equation (2). (a) Here are shown the first (blue), second (black), and third (red) order terms. (b) The second and third order terms are added to the first (blue) and the partial sums are shown ending with the thin white curve showing some steeper slopes near the wave crests. Eventually Stokes predicted the peaks with a steep crest of 120°.

Key Equations

The key equations used for incompressible, inviscid, one-dimensional water waves can summarized as Laplace's equation, a bottom boundary condition and two free surface boundary conditions,

$$\begin{aligned}
\phi_{xx} + \phi_{zz} &= 0 \\
\phi_z &= 0, \quad z = -h, \\
\eta_t + \phi_x \eta_x &= \phi_z, \quad z = \eta(x,t), \\
\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) + gz &= 0, \quad z = \eta(x,t).
\end{aligned} \tag{3}$$

The velocity components are given by $(u, w) = (\phi_x, \phi_z)$.

General Perturbative Analysis

Now, carry out a perturbative analysis of the set of nonlinear equations.

This can lead to solutions for the surface waves or a partial differential equation giving the evolution of the surface waves. [e.g., the KdV Equation, NLS Equation, or higher dimensional nonlinear wave equations].

For example, the Stokes expansion procedure starts by assuming sinusoidal solutions in the form

$$\eta(x,t) = \sum_{n=1}^{\infty} \epsilon^n a_n \cos nk(x-ct)$$

$$\phi(x,z,t) = \sum_{n=1}^{\infty} \epsilon^n b_n(z) \sin nk(x-ct). \tag{4}$$

Then, the expansion coefficients are obtained at various orders of the small parameter ϵ.

General Solution

Since $\phi(x, z, t)$ satisfies Laplace's equation, $b_n''(z) = n^2 k^2 b_n(z)$. The general solution is

$$b_n(z) = A_n \cosh nkz + B_n \sinh nkz.$$

Now, apply the bottom condition, $b_n'(-h) = 0$, to find that

$$b_n(z) = A_n \cosh nk(z + h).$$

So, now we have

$$\phi(x, z, t) = \sum_{n=1}^{\infty} \epsilon^n A_n \cosh nk(z + h) \sin nk(x - ct). \tag{5}$$

Apply Two Surface Boundary Conditions

Insert the series for $\eta(x,t)$ and $\phi(x,z,t)$ and consider the contributions at each order of ϵ.

At leading order, we need the linearized version of the last two equations in system (3). Therefore, for $z = \eta(x,t)$,

$$\begin{aligned} \eta_t &\approx \phi_z, \\ \phi_t + g\eta &\approx 0, \end{aligned} \tag{6}$$

or to leading order

$$\begin{aligned} a_1 kc \sin k(x-ct) &\approx A_1 k \sinh kh \sin k(x-ct), \\ A_1 kc \cosh kh \cos k(x-ct) &\approx g a_1 \cos k(x-ct). \end{aligned} \tag{7}$$

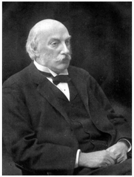
Thus, we have

$$a_1 c = A_1 \sinh kh, \quad A_1 kc \cosh kh = g a_1.$$

So, $A_1 = \dfrac{c}{\sinh kh} a_1$ and $kc^2 = g \tanh kh$.

Stokes went on to compute higher order corrections.

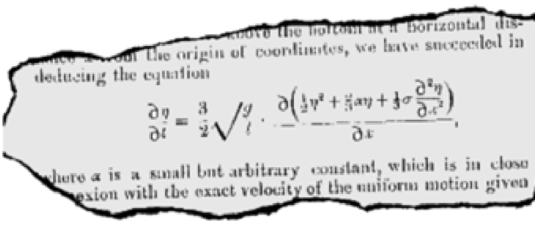
Finding the Balance

Joseph Valentin Boussinesq (1842-1929) and Lord Rayleigh (John William Strutt, 1842-1919)

Diederik Johannes Korteweg (1848-1941) and Gustav de Vries (1866-1934)



the bottom at a horizontal dis-
the origin of coordinates, we have succeeded in deducing the equation

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{l}} \cdot \frac{\partial\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2\eta}{\partial x^2}\right)}{\partial x},$$

here α is a small but arbitrary constant, which is in close
exion with the exact velocity of the uniform motion given

KdV Traveling Waves Solutions

KdV Equation: $u_t + 6uu_x + u_{xxx} = 0$.

Seek solutions of form $u(x, t) = f(x - ct) = f(z)$. Then,

$$\begin{aligned} 0 &= -cf' + 6ff' + f''' \\ 0 &= [-cf + 3f^2 + f'']' \\ A &= -cf + 3f^2 + f'' \\ Af' &= -cff' + 3f^2 f' + f'' f' \\ Af + B &= -\frac{c}{2} f^2 + f^3 + \frac{1}{2}(f')^2 \end{aligned} \tag{8}$$

So,

$$\frac{1}{2}\left(\frac{df}{dz}\right)^2 = B + Af + \frac{c}{2} f^2 - f^3$$

Set $A = B = 0$.

Solitons and Cnoidal Wave Solutions

Solve

$$\left(\frac{df}{dz}\right)^2 = cf^2 - 2f^3.$$

for $\frac{df}{dz}$ and separate variables:

$$dz = \frac{df}{\sqrt{f^2(c-2f)}}.$$

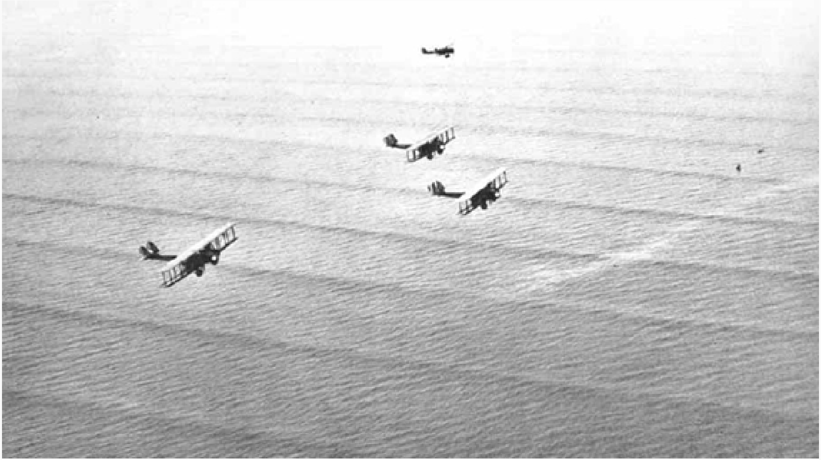
Let $f = \frac{c}{2}\operatorname{sech}^2 u$ and $df = -c\operatorname{sech}^2 u \tanh u\, du$:

$$\begin{aligned} dz &= \frac{-c\operatorname{sech}^2 u \tanh u\, du}{\frac{c}{2}\operatorname{sech}^2 u\sqrt{c - c\operatorname{sech}^2 u}} = -\frac{2}{\sqrt{c}}\, du \\ z - z_0 &= -\frac{2}{\sqrt{c}}\operatorname{sech}^{-2}\frac{2f}{c} \qquad (9) \end{aligned}$$

Then, $u(x,t) = \frac{c}{2}\operatorname{sech}^2\frac{\sqrt{c}}{2}(x - ct - z_0)$. Let $\eta = \frac{\sqrt{c}}{2}$, then

$$u(x,t) = 2\eta^2\operatorname{sech}^2\eta(x - 4\eta^2 t - z_0). \qquad (10)$$

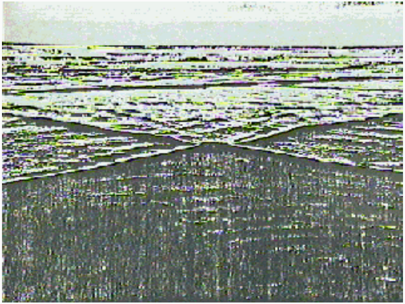
Cnoidal Waves

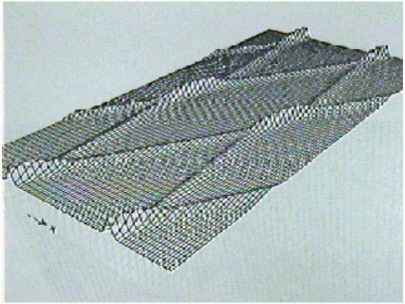


US Army bombers flying over near-periodic swell in shallow water, close to the

Kadomtsev–Petviashvili Equation:

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0$$



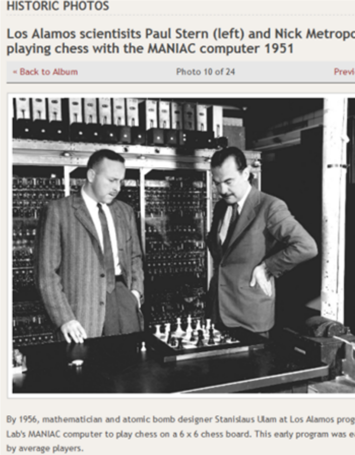


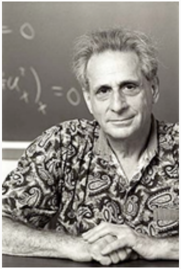
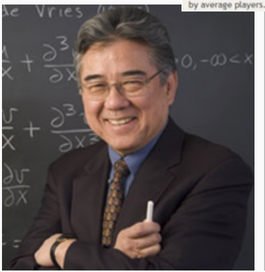
The KdV Resurgence - 1960s

KdV Equation: $u_t + \alpha u u_x + \beta u_{xxx} = 0.$

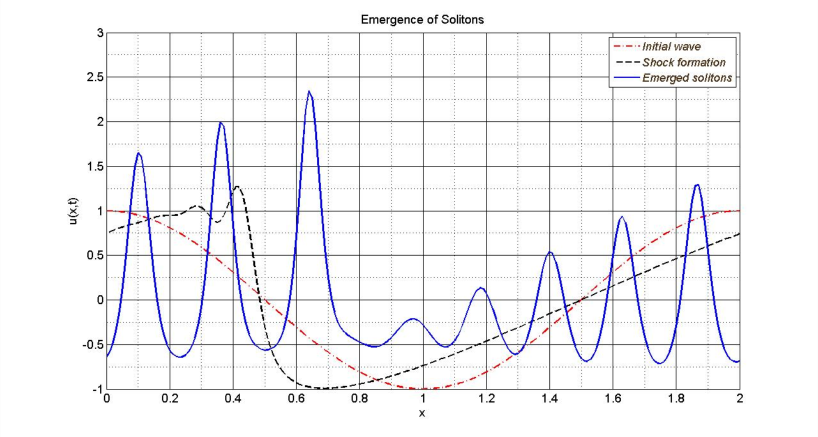
Gardner, Greene, Kruskal, Miura - 1965

- Fermi, Pasta, Ulam (FPU) Problem - 1954.
- Birth of Electronic Computers
- Coined "soliton."
- Started a revolution




The Emergence of Solitons - MATLAB Reconstruction

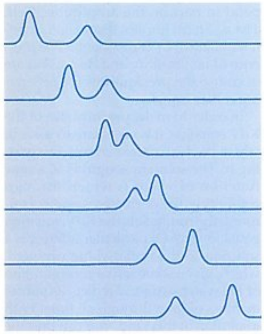


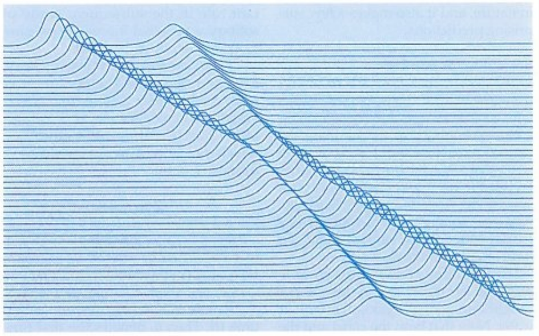
Following the initial condition $u(x, 0) = \cos \pi x$.

A Two Soliton Solution

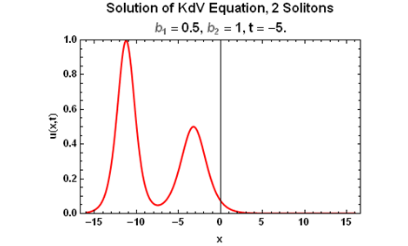
$$u(x,t) = 2(p^2 - q^2)\frac{p^2 \operatorname{csch}^2 \theta + q^2 \operatorname{sech}^2 \chi}{(p \coth \theta - q \tanh \chi)^2}$$

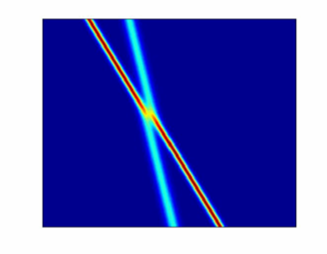
where $\theta = px - 4p^3 t + \theta_0, \chi = qx - 4q^3 t + \chi_0$.

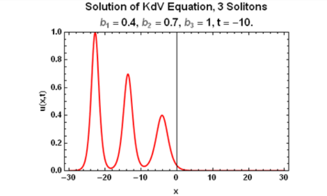




Frame Title


Solution of KdV Equation, 2 Solitons
$b_1 = 0.5$, $b_2 = 1$, t = −5.
u(x,t)
x
1.0
0.8
0.6
0.4
0.2
0.0
−15
−10
−5
0
5
10
15




Solution of KdV Equation, 3 Solitons
$b_1 = 0.4$, $b_2 = 0.7$, $b_3 = 1$, t = −10.
u(x,t)
x
1.0
0.8
0.6
0.4
0.2
0.0
−30
−20
−10
0
10
20
30

Inverse Scattering Transform

Lax Pair: $L\psi = \lambda\psi$, $\psi_t = M\psi$.

Then, $L_t = [L, M]$ is KdV, $u_t - 6uu_x + u_{xxx} = 0$, for $M = -4\partial_x^3 + 6u\partial_x + 3u_x$, and $L = -\partial_x^2 + u$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
u(x,0) \arrow[r, "\text{Direct Scattering}", "\psi'' - u(x,0)\psi = \lambda\psi"'] \arrow[d, dashed, "\text{KdV Evolution}"'] & \begin{array}{c}\text{Scattering Data}\\ S(0), t = 0\end{array} \arrow[d, "\text{Time Evolution}"] \\
u(x,t) & \begin{array}{c}\text{Scattering Data}\\ S(t), t > 0\end{array} \arrow[l, "\text{Inverse Scattering Transform}"]
\end{tikzcd}
$$

Other Nonlinear Evolution Equations

- Nonlinear Schrödinger Equation

$$iu_t + u_{xx} + |u|^2 u = 0.$$

- sine-Gordon Equation

$$u_{tt} - u_{xx} + \sin u = 0.$$

- Extend IST beyond KdV Equation
  - Zakharov-Shabat, 1979
  - Ablowitz, Kaup, Newell, Segur, 1974
- Add perturbations, perturbed KdV

$$u_t + 6uu_x + u_{xxx} = \epsilon F(x, t, u, u_x, ...)$$

Are solitons stable?

Nonlinear Optics

- 1973 Hasegawa Tappert predicted optical solitons and use in communications
- 1987, the first experimental observation of the propagation in an optical fiber.
- 1988, Mollenauer and his team transmitted soliton pulses over 4,000 kilometers.
- 1991, Bell Labs team transmitted solitons error free at 2.5 gigabits over more than 14,000 km.
- 1998, Georges and his team demonstrated a data transmission of 1 terabit per second (1,000,000,000,000 units of info. per sec).
- 2001, practical use of solitons became a reality when Algety Telecom deployed submarine telecommunications equipment in Europe carrying real traffic using Russell's solitary wave.

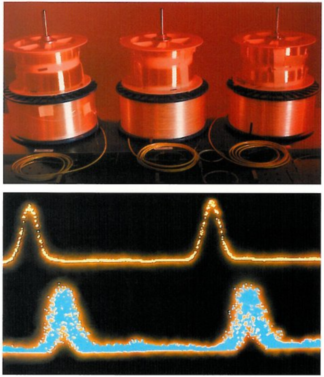
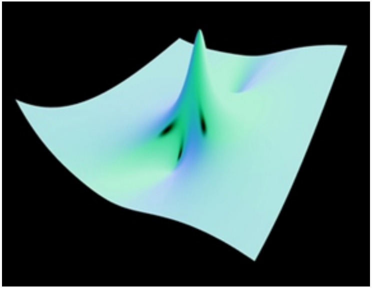


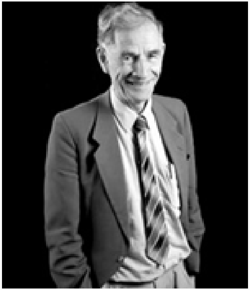
Figure 10. Fiber-optic communications could be enhanced by encoding information in solitons. Investigators at AT&T Bell Laboratories have been experimenting with soliton propagation since they first transmitted one through an optical fiber in 1980. Here the oscilloscope trace of a series of solitons is shown before *(yellow trace)* and after *(blue trace)* traveling 10,000 kilometers in an optical fiber. The pulses show little tendency to dispersion. The transmission rate was five billion bits per second, which would be equivalent to about 100,000 digitized voice channels. (Photograph courtesy of AT&T Bell Laboratories.)

Peregrine Soliton

1983 – Peregrine predicted spatio-temporal evolution of an NLS soliton

20 years later – used as protypical example of rogue waves – in water and in optics.

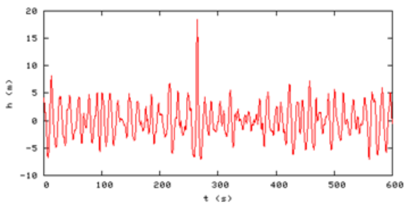




Rogue Waves

Draupner Wave – Jan 1, 1995

- Oil platform in the central North Sea
- Minor damage
- Read by a laser sensor.
- During wave heights of 12 m (39ft),
  - Freak wave - max height of 25.6 m (84ft)
  - (peak elevation was 18.5 m (61ft)).
- Estimated – 1 in 200,000 wave (P. Taylor).



Mathematical Rogue Waves

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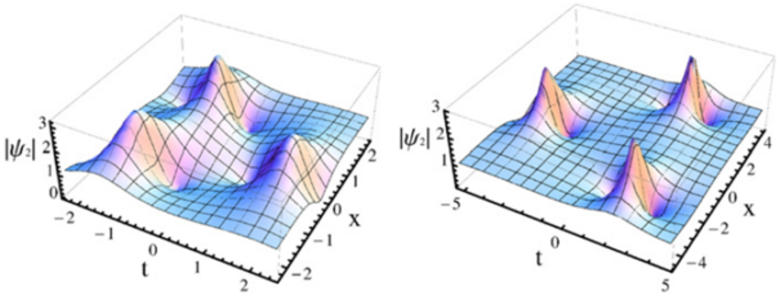

Fig. 3. Rogue wave triplets. Parameters (a) $\gamma = 20$ and $\beta = 40$; (b) $\gamma = 100$ and $\beta = -400$.

Summary