Geodesic Equations for the Wormhole Metric

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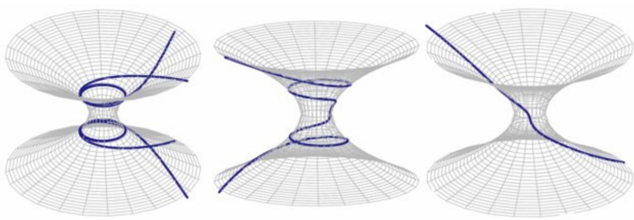
The Wormhole Metric

Morris and Thorne wormhole metric: [M. S. Morris, K. S. Thorne, Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity, *Am. J. Phys.* **56**, 395-412, 1988.]

$$\boxed{ds^2 = -c^2 dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2\theta\, d\phi^2)}$$



Embedding $ds^2 = -dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2\theta\, d\phi^2)$

Consider 2D slices (t =const, $\theta = \pi/2$). Then,

$$dS^2 = dr^2 + (b^2 + r^2)\, d\phi^2.$$

Compare to a cylindrical coordinate line element: $(\rho(r), \psi, z(r))$

$$\begin{aligned} d\Sigma^2 &= d\rho^2 + \rho^2\, d\psi^2 + dz^2 \\ &= \left[\left(\frac{dz}{dr}\right)^2 + \left(\frac{d\rho}{dr}\right)^2\right] dr^2 + \rho^2(r)\, d\phi^2. \end{aligned}$$

Then, $\rho^2 = r^2 + b^2$ and $\left(\frac{dz}{dr}\right)^2 + \left(\frac{d\rho}{dr}\right)^2 = 1$.

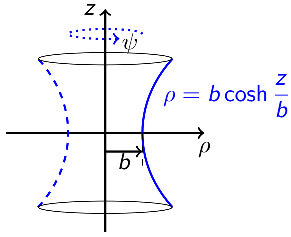
Since $\rho\, d\rho = r\, dr$, $\frac{d\rho}{dr} = \frac{r}{\rho} = \frac{r}{\sqrt{r^2 + b^2}}$. Therefore,

$$\left(\frac{dz}{dr}\right)^2 = 1 - \frac{r^2}{r^2 + b^2} = \frac{b^2}{r^2 + b^2}.$$

Embedding Diagram from $\left(\frac{dz}{dr}\right)^2 = \frac{b^2}{r^2+b^2}$

Now we integrate [substitute $r = b\sinh u$, $dr = b\cosh u\, du$]:

$$\begin{aligned} \frac{dz}{dr} &= \frac{b}{\sqrt{b^2 + r^2}} \\ z &= b\int \frac{dr}{\sqrt{b^2 + r^2}} \\ &= b\int \frac{b\cosh u\, du}{\sqrt{b^2(1 + \sinh^2 u)}} \end{aligned}$$



Therefore, $z = bu = b\sinh^{-1}\frac{r}{b}$, or

$$\rho = b\cosh\frac{z}{b}.$$

The embedded surface of revolution is a hyperboloid.

Lagrangian Approach to Geodesics

Begin with a metric $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$. Then,

$$\tau_{AB} = \int_0^1 \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}}\, d\sigma.$$

Euler-Lagrange Equations $\Rightarrow$ Geodesic Equations

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{x}^\gamma}\right) - \frac{\partial L}{\partial x^\gamma} = 0, \qquad \gamma = 0, 1, 2, 3,$$

where $\dot{x}^\gamma = \frac{dx^\gamma}{d\sigma}$ and we defined the "Lagrangian"

$$L(x^\gamma, \dot{x}^\gamma) = \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}} = \frac{d\tau}{d\sigma}.$$

Compute $\frac{\partial L}{\partial x^\gamma} - \frac{d}{d\sigma}\left(\frac{\partial L}{\partial(dx^\gamma/d\sigma)}\right) = 0.$

We carefully compute the derivatives for a general metric.

$$\begin{aligned} \frac{\partial L}{\partial x^\gamma} &= -\frac{1}{2L}\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma} \\ &= -\frac{L}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}. \\ \frac{\partial L}{\partial(dx^\gamma/d\sigma)} &= -\frac{1}{2L}g_{\alpha\beta}\left(\delta^\alpha_\gamma\frac{dx^\beta}{d\sigma} + \frac{dx^\alpha}{d\sigma}\delta^\beta_\gamma\right) \\ &= -\frac{1}{2L}\left(g_{\gamma\beta}\frac{dx^\beta}{d\sigma} + g_{\alpha\gamma}\frac{dx^\alpha}{d\sigma}\right) \\ &= -\frac{1}{L}g_{\alpha\gamma}\frac{dx^\alpha}{d\sigma}. \end{aligned}$$

The σ derivatives have been replaced by $\frac{df}{d\sigma} = \frac{df}{d\tau}\frac{d\tau}{d\sigma} = L\frac{df}{d\tau}$. We used symmetry and the fact that α and β are dummy indices.

Compute $\frac{\partial L}{\partial x^\gamma} - \frac{d}{d\sigma}\left(\frac{\partial L}{\partial(dx^\gamma/d\sigma)}\right) = 0$. (cont'd)

We differentiate the last result:

$$\begin{aligned}
-\frac{d}{d\sigma}\left(\frac{\partial L}{\partial(dx^\gamma/d\sigma)}\right) &= \frac{d}{d\sigma}\left(\frac{1}{L}g_{\alpha\gamma}\frac{dx^\alpha}{d\sigma}\right) \\
&= L\frac{d}{d\tau}\left(g_{\alpha\gamma}\frac{dx^\alpha}{d\tau}\right) \\
&= L\left[g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2} + \frac{dg_{\alpha\gamma}}{d\tau}\frac{dx^\alpha}{d\tau}\right] \\
&= L\left[g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2} + \frac{dg_{\alpha\gamma}}{dx^\beta}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}\right] \\
&= L\left[g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2} + \frac{1}{2}\left(\frac{dg_{\alpha\gamma}}{dx^\beta} + \frac{dg_{\gamma\beta}}{dx^\alpha}\right)\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}\right].
\end{aligned}$$

We have used symmetry, re-indexing of repeated indices, and have eliminated appearances of L.

Compute $\frac{\partial L}{\partial x^\gamma} - \frac{d}{d\sigma}\left(\frac{\partial L}{\partial(dx^\gamma/d\sigma)}\right) = 0$. (finally!)

So far, we have

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x^\gamma} - \frac{d}{d\sigma}\left(\frac{\partial L}{\partial(dx^\gamma/d\sigma)}\right) \\
&= L\left[g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2} + \frac{1}{2}\left(\frac{dg_{\alpha\gamma}}{dx^\beta} + \frac{dg_{\gamma\beta}}{dx^\alpha}\right)\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}\right] - \frac{L}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}.
\end{aligned}$$

Rearranging the terms and changing the dummy index α to δ ,

$$\begin{aligned}
g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2} &= \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} - \frac{1}{2}\left(\frac{dg_{\alpha\gamma}}{dx^\beta} + \frac{dg_{\gamma\beta}}{dx^\alpha}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \\
&= -\frac{1}{2}\left[\frac{dg_{\alpha\gamma}}{dx^\beta} + \frac{dg_{\gamma\beta}}{dx^\alpha} - \frac{dg_{\alpha\beta}}{dx^\gamma}\right]\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \\
&= -\frac{1}{2}\left[\frac{dg_{\delta\gamma}}{dx^\beta} + \frac{dg_{\gamma\beta}}{dx^\delta} - \frac{dg_{\delta\beta}}{dx^\gamma}\right]\frac{dx^\delta}{d\tau}\frac{dx^\beta}{d\tau} \\
&\equiv -g_{\alpha\gamma}\Gamma^\alpha_{\delta\beta}\frac{dx^\delta}{d\tau}\frac{dx^\beta}{d\tau}.
\end{aligned}$$

The Result: Key Equations

The Geodesic Equations

$$\boxed{\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0,}$$

In terms of the four-velocity:

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma = 0.$$

The Christoffel Symbols are given by [Note: $\Gamma^\delta_{\beta\gamma} = \Gamma^\delta_{\gamma\beta}$.]

$$\boxed{g_{\alpha\delta}\Gamma^\delta_{\beta\gamma} = \frac{1}{2}\left[\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right],}$$

Wormhole Geodesics via the Lagrangian

Begin with the proper time (with $c = 1$),

$$d\tau^2 = -ds^2 = dt^2 - dr^2 - (b^2 + r^2)(d\theta^2 + \sin^2\theta\, d\phi^2),$$

Write the Lagrangian,

$$L = \sqrt{\left(\frac{dt}{d\sigma}\right)^2 - \left(\frac{dr}{d\sigma}\right)^2 - (b^2 + r^2)\left(\left(\frac{d\theta}{d\sigma}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\sigma}\right)^2\right)},$$

Apply the Euler-Lagrange equation for each variable: t, r, θ, ϕ.

Example - time variable t, $\quad \dot{t} \equiv \frac{dt}{d\sigma}$:

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{t}}\right) - \frac{\partial L}{\partial t} = 0.$$

Time Equation

Lagrangian:

$$L = \left[\dot{t}^2 - \dot{r}^2 - (b^2 + r^2)(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\right]^{1/2}$$

Geodesic Equation for t: [Recall that $L\frac{d}{d\tau} = \frac{d}{d\sigma}$]

$$\begin{aligned}
\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{t}}\right) &= \frac{\partial L}{\partial t} \\
\frac{d}{d\sigma}\left(\frac{\cancel{2}}{\cancel{2}L}\frac{dt}{d\sigma}\right) &= 0 \\
L\frac{d}{d\tau}\left(\frac{dt}{d\tau}\right) &= 0
\end{aligned}$$

$$\boxed{\frac{d^2t}{d\tau^2} = 0.}$$

Radial Equation

Lagrangian:

$$L = \left[\dot{t}^2 - \dot{r}^2 - (b^2 + r^2)(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\right]^{1/2}$$

Geodesic Equation for r:

$$\begin{aligned}
\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{r}}\right) &= \frac{\partial L}{\partial r} \\
L\frac{d}{d\tau}\left(-\frac{1}{L}\frac{dr}{d\sigma}\right) &= -\frac{1}{2L}(2r)\left[\left(\frac{d\theta}{d\sigma}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\sigma}\right)^2\right]
\end{aligned}$$

$$\boxed{\frac{d^2r}{d\tau^2} = r\left[\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right].}$$

The θ-Equation

Lagrangian:

$$L = \left[\dot{t}^2 - \dot{r}^2 - (b^2 + r^2)(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\right]^{1/2}$$

Geodesic Equation for θ:

$$\begin{aligned} \frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= \frac{\partial L}{\partial \theta} \\ L\frac{d}{d\tau}\left(-\frac{b^2+r^2}{L}\frac{d\theta}{d\sigma}\right) &= -\frac{1}{2L}(b^2+r^2)\left[2\sin\theta\cos\theta\left(\frac{d\phi}{d\sigma}\right)^2\right] \end{aligned}$$

$$\boxed{\frac{d}{d\tau}\left((b^2+r^2)\frac{d\theta}{d\tau}\right) = (b^2+r^2)\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2.}$$

Christoffel Symbols from the Geodesic Equations

Start with general Geodesic Equation:

$$\boxed{\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}}$$

$$\begin{aligned} \frac{d^2t}{d\tau^2} &= 0. \\ \frac{d^2r}{d\tau^2} &= r\left[\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right]. \end{aligned}$$

Read off the coefficients:

- $\Gamma^t_{\beta\gamma} = 0, \quad \beta,\gamma = r,\theta,\phi.$
- $\Gamma^r_{\theta\theta} = -r$, $\Gamma^r_{\phi\phi} = -r\sin^2\theta.$

The ϕ-Equation

Lagrangian:

$$L = \left[\dot{t}^2 - \dot{r}^2 - (b^2 + r^2)(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\right]^{1/2}$$

Geodesic Equation for ϕ:

$$\begin{aligned} \frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{\phi}}\right) &= \frac{\partial L}{\partial \phi} \\ L\frac{d}{d\tau}\left(-\frac{b^2+r^2}{L}\sin^2\theta\frac{d\phi}{d\sigma}\right) &= 0 \end{aligned}$$

$$\boxed{\frac{d}{d\tau}\left((b^2+r^2)\sin^2\theta\frac{d\phi}{d\tau}\right) = 0.}$$

Christoffel Symbols (cont'd)

$$\boxed{\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}}$$

$$\begin{aligned} \frac{d}{d\tau}\left((b^2+r^2)\frac{d\theta}{d\tau}\right) &= (b^2+r^2)\sin\theta\cos\theta\left(\frac{d\phi}{d\sigma}\right)^2 \\ (b^2+r^2)\frac{d^2\theta}{d\tau^2} + 2r\frac{dr}{d\tau}\frac{d\theta}{d\tau} &= (b^2+r^2)\sin\theta\cos\theta\left(\frac{d\phi}{d\sigma}\right)^2 \\ \frac{d^2\theta}{d\tau^2} &= -\frac{2r}{b^2+r^2}\frac{dr}{d\tau}\frac{d\theta}{d\tau} + \sin\theta\cos\theta\left(\frac{d\phi}{d\sigma}\right)^2. \end{aligned}$$

- $\Gamma^\theta_{\theta r} = \frac{r}{b^2+r^2} = \Gamma^\theta_{r\theta}$, $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta.$
- Note: $\Gamma^\theta_{\theta r}$ and $\Gamma^\theta_{r\theta}$ contribute equally, thus there is no 2.

The Geodesic Equations for the MT Wormhole

$$\begin{aligned} \frac{d^2t}{d\tau^2} &= 0 \\ \frac{d^2r}{d\tau^2} &= r\left[\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right] \\ \frac{d}{d\tau}\left((b^2+r^2)\frac{d\theta}{d\tau}\right) &= (b^2+r^2)\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 \\ \frac{d}{d\tau}\left((b^2+r^2)\sin^2\theta\frac{d\phi}{d\tau}\right) &= 0. \end{aligned}$$

- Solve for geodesics $(t(\tau), r(\tau), \theta(\tau), \phi(\tau))$.
- Read off Christoffel Symbols, $\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}$

Christoffel Symbols (cont'd)

$$\boxed{\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}}$$

$$\begin{aligned} \frac{d}{d\tau}\left((b^2+r^2)\sin^2\theta\frac{d\phi}{d\tau}\right) &= 0. \\ (b^2+r^2)\sin^2\theta\frac{d^2\phi}{d\tau} &= -2r\sin^2\theta\frac{dr}{d\tau}\frac{d\phi}{d\tau} \\ &\quad -2(b^2+r^2)\sin\theta\cos\theta\frac{d\theta}{d\tau}\frac{d\phi}{d\tau}. \\ \frac{d^2\phi}{d\tau} &= -\frac{2r}{b^2+r^2}\frac{dr}{d\tau}\frac{d\phi}{d\tau} - 2\cot\theta\frac{d\theta}{d\tau}\frac{d\phi}{d\tau}. \end{aligned}$$

- $\Gamma^\phi_{\phi r} = \frac{r}{b^2+r^2} = \Gamma^\phi_{r\phi}$, $\Gamma^\phi_{\phi\theta} = \cot\theta.$

Christoffel Symbols from the Metric

The Christoffel symbols are defined by

$$\boxed{g_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} = \frac{1}{2}\left[\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right].}$$

For the wormhole metric,

$$ds^2 = -dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2\theta\, d\phi^2).$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b^2 + r^2 & 0 \\ 0 & 0 & 0 & (b^2 + r^2)\sin^2\theta \end{pmatrix},$$

or, $g_{tt} = -1$, $g_{rr} = 1$, $g_{\theta\theta} = b^2 + r^2$, $g_{\phi\phi} = (b^2 + r^2)\sin^2\theta$.

Christoffel Symbols $\Gamma^{t}_{\beta\gamma}$

The nonzero metric elements are
$g_{tt} = -1$, $g_{rr} = 1$, $g_{\theta\theta} = b^2 + r^2$, $g_{\phi\phi} = (b^2 + r^2)\sin^2\theta$.
Let $\alpha = t$ and $x^{\alpha} = t$, then

$$g_{t\delta}\Gamma^{\delta}_{\beta\gamma} = \frac{1}{2}\left[\frac{\partial g_{t\beta}}{\partial x^{\gamma}} + \frac{\partial g_{t\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial t}\right].$$

Since the $g_{t\mu}$ is nonzero and constant for $\mu = t$,

$$\begin{aligned} g_{tt}\Gamma^{t}_{\beta\gamma} &= \frac{1}{2}\left[\frac{\partial g_{t\beta}}{\partial x^{\gamma}} + \frac{\partial g_{t\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial t}\right] \\ g_{tt}\Gamma^{t}_{tt} &= \frac{1}{2}\left[\frac{\partial g_{tt}}{\partial t} + \frac{\partial g_{tt}}{\partial t} - \frac{\partial g_{tt}}{\partial t}\right] = 0. \end{aligned} \tag{1}$$

So, $\Gamma^{t}_{\alpha\beta} = 0$ for all α and β.

Christoffel Symbols $\Gamma^{r}_{\alpha\beta}$

The metric elements are
$g_{tt} = -1$, $g_{rr} = 1$, $g_{\theta\theta} = b^2 + r^2$, $g_{\phi\phi} = (b^2 + r^2)\sin^2\theta$.
Let $\alpha = r$ and $x^{\alpha} = r$, then

$$g_{r\delta}\Gamma^{\delta}_{\beta\gamma} = \frac{1}{2}\left[\frac{\partial g_{r\beta}}{\partial x^{\gamma}} + \frac{\partial g_{r\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial r}\right].$$

Thus, $\delta = r$ and either $\beta = \gamma = \theta$ or $\beta = \gamma = \phi$. So, we have

$$g_{rr}\Gamma^{r}_{\theta\theta} = \frac{1}{2}\left[\frac{\partial g_{r\theta}}{\partial \theta} + \frac{\partial g_{r\theta}}{\partial \theta} - \frac{\partial g_{\theta\theta}}{\partial r}\right].$$

$$g_{rr}\Gamma^{r}_{\phi\phi} = \frac{1}{2}\left[\frac{\partial g_{r\phi}}{\partial \phi} + \frac{\partial g_{r\phi}}{\partial \phi} - \frac{\partial g_{\phi\phi}}{\partial r}\right].$$

Therefore, since $g_{rr} = 1$,

$$\Gamma^{r}_{\theta\theta} = -r, \quad \Gamma^{r}_{\phi\phi} = -r\sin^2\theta.$$

Christoffel Symbols $\Gamma^{\theta}_{\alpha\beta}$

The metric elements are
$g_{tt} = -1$, $g_{rr} = 1$, $g_{\theta\theta} = b^2 + r^2$, $g_{\phi\phi} = (b^2 + r^2)\sin^2\theta$.
Let $\alpha = \theta$ and $x^{\alpha} = \theta$, then

$$g_{\theta\delta}\Gamma^{\delta}_{\beta\gamma} = \frac{1}{2}\left[\frac{\partial g_{\theta\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\theta\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial \theta}\right].$$

Thus, $\delta = \theta$. We take $\beta = \theta$ or $\beta = \phi$ due to symmetry. So, we have

$$g_{\theta\theta}\Gamma^{\theta}_{\theta\gamma} = \frac{1}{2}\left[\frac{\partial g_{\theta\theta}}{\partial x^{\gamma}} + \frac{\partial g_{\theta\gamma}}{\partial \theta} - \frac{\partial g_{\theta\gamma}}{\partial \theta}\right].$$

$$g_{\theta\theta}\Gamma^{\theta}_{\phi\gamma} = \frac{1}{2}\left[\frac{\partial g_{\theta\phi}}{\partial x^{\gamma}} + \frac{\partial g_{\theta\gamma}}{\partial \phi} - \frac{\partial g_{\phi\gamma}}{\partial \theta}\right].$$

Nonzero terms occur for $\gamma = r$ in first and $\gamma = \phi$ in second equation.

Christoffel Symbols $\Gamma^{\theta}_{\alpha\beta}$ (cont'd)

Since $g_{\theta\theta} = b^2 + r^2$ and $g_{\phi\phi} = (b^2 + r^2)\sin^2\theta$, we have

$$\begin{aligned} g_{\theta\theta}\Gamma^{\theta}_{\theta r} &= \frac{1}{2}\left[\frac{\partial g_{\theta\theta}}{\partial r} + \frac{\partial g_{\theta r}}{\partial \theta} - \frac{\partial g_{\theta r}}{\partial \theta}\right]. \\ (b^2 + r^2)\Gamma^{\theta}_{\theta r} &= \frac{1}{2}\frac{\partial g_{\theta\theta}}{\partial r} = r \\ \Gamma^{\theta}_{\theta r} &= \frac{r}{b^2 + r^2} = \Gamma^{\theta}_{r\theta}. \end{aligned}$$

and

$$\begin{aligned} g_{\theta\theta}\Gamma^{\theta}_{\phi\phi} &= \frac{1}{2}\left[\frac{\partial g_{\theta\phi}}{\partial \phi} + \frac{\partial g_{\theta\phi}}{\partial \phi} - \frac{\partial g_{\phi\phi}}{\partial \theta}\right]. \\ (b^2 + r^2)\Gamma^{\theta}_{\phi\phi} &= -\frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial \theta} = -(b^2 + r^2)\sin\theta\cos\theta \\ \Gamma^{\theta}_{\phi\phi} &= -\sin\theta\cos\theta. \end{aligned}$$

Christoffel Symbols $\Gamma^{\phi}_{\alpha\beta}$

The metric elements are
$g_{tt} = -1$, $g_{rr} = 1$, $g_{\theta\theta} = b^2 + r^2$, $g_{\phi\phi} = (b^2 + r^2)\sin^2\theta$.
Let $\alpha = \phi$ and $x^{\alpha} = \phi$, then

$$g_{\phi\delta}\Gamma^{\delta}_{\beta\gamma} = \frac{1}{2}\left[\frac{\partial g_{\phi\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\phi\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial \phi}\right].$$

Thus, $\delta = \phi$ and we take $\beta = \phi$ due to symmetry. So, we have

$$\begin{aligned} g_{\phi\phi}\Gamma^{\phi}_{\phi\gamma} &= \frac{1}{2}\left[\frac{\partial g_{\phi\phi}}{\partial x^{\gamma}} + \frac{\partial g_{\phi\gamma}}{\partial \phi} - \frac{\partial g_{\phi\gamma}}{\partial \phi}\right] \\ &= \frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial x^{\gamma}}. \end{aligned}$$

Since $g_{\phi\phi} = (b^2 + r^2)\sin^2\theta$, then $\gamma = r$ or $\gamma = \theta$.

Christoffel Symbols $\Gamma^{\phi}_{\alpha\beta}$ (cont'd)

Since $g_{\phi\phi} = (b^2 + r^2)\sin^2\theta$, we have

$$\begin{aligned} g_{\phi\phi}\Gamma^{\phi}_{\phi r} &= \frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial r}. \\ (b^2+r^2)\sin^2\theta\,\Gamma^{\phi}_{\phi r} &= r\sin^2\theta \\ g_{\phi\phi}\Gamma^{\phi}_{\phi\theta} &= \frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial\theta}. \\ (b^2+r^2)\sin^2\theta\,\Gamma^{\phi}_{\phi\theta} &= (b^2+r^2)\sin\theta\cos\theta \end{aligned}$$

Therefore, we have

$$\Gamma^{\phi}_{\phi r} = \frac{r}{b^2+r^2} = \Gamma^{\phi}_{r\phi}, \quad \Gamma^{\phi}_{\phi\theta} = \cot\theta = \Gamma^{\phi}_{\theta\phi}.$$

Example: Spherical Coordinates

Transformation:

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta. \end{aligned}$$

Christoffel Symbols

$$\begin{aligned} \Gamma^{r}_{r\theta} &= \frac{\partial^2 x}{\partial r\partial\theta}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r\partial\theta}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial r\partial\theta}\frac{\partial r}{\partial z} \\ &= \cos\theta\cos\phi\frac{x}{r} + \cos\theta\sin\phi\frac{y}{r} - \sin\theta\frac{z}{r} = 0. \\ \Gamma^{r}_{\theta\theta} &= \frac{\partial^2 x}{\partial^2\theta}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial^2\theta}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial^2\theta}\frac{\partial r}{\partial z} \\ &= -r\sin\theta\cos\phi\frac{x}{r} + r\sin\theta\sin\phi\frac{y}{r} - r\cos\theta\frac{z}{r} = -r. \end{aligned}$$

etc.

Wormhole Metric and Geodesic Equations

$$\boxed{ds^2 = -dt^2 + dr^2 + (b^2+r^2)(d\theta^2 + \sin^2\theta\, d\phi^2).}$$

$$\begin{aligned} \frac{d^2t}{d\tau^2} &= 0. \\ \frac{d^2r}{d\tau^2} &= r\left[\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right]. \\ \frac{d}{d\tau}\left((b^2+r^2)\frac{d\theta}{d\tau}\right) &= (b^2+r^2)\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2. \\ \frac{d}{d\tau}\left((b^2+r^2)\sin^2\theta\frac{d\phi}{d\tau}\right) &= 0. \end{aligned}$$

Christoffel Symbols $\Gamma^{r}_{\theta\theta} = -r$, $\Gamma^{r}_{\phi\phi} = -r\sin^2\theta$, $\Gamma^{\theta}_{\theta r} = \frac{r}{b^2+r^2} = \Gamma^{\theta}_{r\theta}$, $\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$, $\Gamma^{\phi}_{\phi r} = \frac{r}{b^2+r^2} = \Gamma^{\phi}_{r\phi}$, $\Gamma^{\phi}_{\phi\theta} = \cot\theta = \Gamma^{\phi}_{\theta\phi}$.

Computing Christoffel Symbols in Maple

```
> restart: with( tensor ):
Declare coordinates in desired order.
> coord := [t, r, theta, phi]:
Enter metric components to produce g.
> gg:=array(symmetric,sparse,1..4,1..4):
  gg[1,1] := -1:  gg[2,2] := 1:      gg[3,3] := r^2+b^2:    gg[4,4] := (r^2+b^2)*sin(theta)^2:
> g := create( [-1,-1], eval(gg));
```

$$g := \text{table}([\mathit{compts} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2+b^2 & 0 \\ 0 & 0 & 0 & (r^2+b^2)\sin(\theta)^2 \end{bmatrix}, \mathit{index\_char} = [-1, -1]])$$

```
Run main routine and display Chritoffel symbols (of second kind).
> tensorsGR(coord,g,contra_metric,det_met, C1, C2, Rm, Rc, R, G, C):
> displayGR(Christoffel2,C2);
```

The Christoffel Symbols of the Second Kind

non-zero components :

$$\{2,33\} = -r$$

$$\{2,44\} = -r\sin(\theta)^2$$

$$\{3,23\} = \frac{r}{r^2+b^2}$$

$$\{3,44\} = -\sin(\theta)\cos(\theta)$$

$$\{4,24\} = \frac{r}{r^2+b^2}$$

$$\{4,34\} = \frac{\cos(\theta)}{\sin(\theta)}$$

Computing $\Gamma^{\alpha}_{\beta\gamma}$ without Lagrangians

Let's compare $\Gamma^{\alpha'}_{\beta'\gamma'}$ in basis $x^{\alpha'}$ to $\Gamma^{\alpha}_{\beta\gamma}$ in basis x^{α}. For $x^{\alpha} = x^{\alpha}(x^{\mu'})$ we define

$$L^{\alpha}_{\mu'} = \frac{\partial x^{\alpha}}{\partial x^{\mu'}}.$$

Then, we have (MTW, p. 262),

$$\Gamma^{\alpha'}_{\beta'\gamma'} = L^{\alpha'}_{\rho} L^{\mu}_{\beta'} L^{\nu}_{\gamma'} \Gamma^{\rho}_{\mu\nu} + L^{\alpha'}_{\mu} L^{\mu}_{\beta',\gamma'},$$

where the bases are $e_{\mu'} = L^{\alpha}_{\mu'} e_{\alpha}$ at a given point.
So,

$$\Gamma^{\alpha'}_{\beta'\gamma'} = \frac{\partial x^{\alpha'}}{\partial x^{\rho}}\frac{\partial x^{\mu}}{\partial x^{\beta'}}\frac{\partial x^{\nu}}{\partial x^{\gamma'}}\Gamma^{\rho}_{\mu\nu} + \frac{\partial x^{\alpha'}}{\partial x^{\mu}}\frac{\partial^2 x^{\mu}}{\partial x^{\gamma'}\partial x^{\beta'}}.$$

Since the Christoffel symbols vanish for flat coordinates,

$$\Gamma^{\alpha'}_{\beta'\gamma'} = \frac{\partial x^{\alpha'}}{\partial x^{\mu}}\frac{\partial^2 x^{\mu}}{\partial x^{\gamma'}\partial x^{\beta'}}.$$

For example, to find $\Gamma^{\alpha'}_{\beta'\gamma'}$ for spherical coordinates, one computse a few derivatives of the spherical coordinate transformations.