Name: ____ **Score:** ☐

Problem 1. Pick out the expressions which are written properly. If not proper, then explain. We do not care if they are true or false.

1. $g_{\alpha\beta}dx^\alpha dx^\beta = g_{\alpha\beta}dx^\alpha dx^\gamma$
2. $g_{\alpha\beta}a^\alpha b^\beta = g_{\alpha\delta}a^\alpha b^\delta$
3. $\Gamma^\alpha_{\beta\gamma}a^\alpha b^\beta c^\gamma = d^\alpha$
4. $g_{\alpha\beta}\dfrac{\partial x^\alpha}{\partial x'^\gamma}\dfrac{\partial x^\beta}{\partial x'^\delta} = g_{\gamma\delta}\dfrac{\partial x^\gamma}{\partial x'^\alpha}\dfrac{\partial x^\delta}{\partial x'^\beta}$
5. $\dfrac{\partial g_{\alpha\beta}}{\partial x^\gamma} = 0$
6. $\Gamma^\alpha_{\alpha\beta} = \Gamma^\beta_{\beta\beta}$

Problem 2. Let T^α be a contravariant vector and S_α be a covariant vector.

a. Show that $R_\beta = g_{\alpha\beta}T^\alpha$ is a covariant vector.

b. Show that $R^\beta = g^{\alpha\beta}S_\alpha$ is a contravariant vector.

Problem 3. Show that $T^{\alpha\beta\gamma\delta\rho}S_{\beta\rho}$ is a tensor. What is its rank?

Problem 4. Recall that $g_{\alpha\beta} = g_{\beta\alpha}$ and $\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}$.

a. For three dimensional space, how many different values are there for g and Γ?

b. For a spacetime with one temporal and three spatial dimensions, how many different values are there for g and Γ?

Problem 5. The line element in terms of the metric tensor, $g_{\alpha\beta}$ is given by

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta.$$

Show that the transformed metric for the transformation $x'^\alpha = x'^\alpha(x^\beta)$ is given by

$$g'_{\gamma\delta} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\gamma} \frac{\partial x^\beta}{\partial x'^\delta}$$

Problem 6. Consider cylindrical coordinates,

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z, \end{aligned} \tag{1}$$

with the basis $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$, and $\hat{\mathbf{e}}_z$.

a. Let $\mathbf{V} = V^r\hat{\mathbf{e}}_r + V^\theta\hat{\mathbf{e}}_\theta + V^z\hat{\mathbf{e}}_z$. Find $\frac{\partial \mathbf{V}}{\partial r}, \frac{\partial \mathbf{V}}{\partial \theta}, \frac{\partial \mathbf{V}}{\partial z}$. [Hint: Similar to the polar coordinate example, write this basis in terms of $\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z$, in order to write derivatives in terms of the cylindrical basis.]

b. From the results in part a, determine all of the Christoffel symbols, $\Gamma^r_{rr}, \Gamma^r_{r\theta}, \Gamma^\theta_{rr}$, etc.

Problem 7. Do the following exercises from the text: 1.1, 1.3, 1.4

Problem 8. Consider the metric on S^2 (surface of sphere),

$$dS^2 = a^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) .$$

a. Calculate the Christoffel symbols.

b. Find the geodesic equations.

c. Use the geodesic equations to show that a great circle is a solution by orienting the coordinates to make the computation simple.