I. $\frac{d^2y}{dt^2} + 2y = 0 \Rightarrow r^2 + 2 = 0 \Rightarrow r = \pm\sqrt{2}\, i$

$\therefore$ in both (a) & (b) $y_h(t) = c_1 \cos\sqrt{2}\, t + c_2 \sin\sqrt{2}\, t$

(a) $\omega \neq \sqrt{2}$, $y(0) = 1$, $y'(0) = 0$, $\frac{d^2y}{dt^2} + 2y = \epsilon \sin\omega t$

By undetermined coefficients, $y_p(t) = a_1 \cos\omega t + a_2 \sin\omega t$

$$y_p''(t) = \omega^2 a_1(-\cos\omega t) + \omega^2 a_2(-\sin\omega t)$$
$$= -\omega^2[a_1\cos\omega t + a_2 \sin\omega t]$$
$$= -\omega^2 y_p(t)$$

$\therefore \frac{d^2y}{dt^2} + 2y = (2-\omega^2)(a_1\cos\omega t + a_2\sin\omega t) = \epsilon \sin\omega t$

$\therefore (2-\omega^2)a_1 = 0 \Rightarrow a_1 = 0$

$(2-\omega^2)a_2 = \epsilon \Rightarrow a_2 = \frac{\epsilon}{2-\omega^2}$

$\therefore y_p(t) = \frac{\epsilon}{2-\omega^2}\sin\omega t$

$y(t) = \frac{\epsilon}{2-\omega^2}\sin\omega t + c_1\cos\sqrt{2}\, t + c_2 \sin\sqrt{2}\, t$

$1 = y(0) = c_1$

$y'(t) = \frac{\epsilon\omega}{2-\omega^2}\cos\omega t - \sqrt{2}\, c_1 \sin\sqrt{2}\, t + \sqrt{2}\, c_2 \cos\sqrt{2}\, t$

$1 = y'(0) = \frac{\epsilon\omega}{2-\omega^2} + \sqrt{2}\, c_2 \Rightarrow c_2 = \frac{1 - \frac{\epsilon\omega}{2-\omega^2}}{\sqrt{2}} = \frac{2-\omega^2-\epsilon\omega}{\sqrt{2}(2-\omega^2)}$

$$\boxed{y(t) = \frac{\epsilon}{2-\omega^2}\sin\omega t + \cos\sqrt{2}\, t + \frac{2-\omega^2-\epsilon\omega}{\sqrt{2}(2-\omega^2)}\sin\sqrt{2}\, t}$$

$\sqrt{2} = \omega$ (b) By undetermined coefficients, $y_p(t) = a_1 t\cos\omega t + a_2 t \sin\omega t$

$$y_p'(t) = a_1\cos\omega t + a_2 \sin\omega t - \omega a_1 t \sin\omega t + \omega a_2 t \cos\omega t$$
$$= a_1 \cos\omega t + a_2\sin\omega t + \omega t(-a_1 \sin\omega t + a_2 \cos\omega t)$$
$$= (a_1 + a_2\omega t)\cos\omega t + (a_2 - a_1\omega t)\sin\omega t$$

$$y_p''(t) = [a_2\omega + \omega(a_2 - a_1\omega t)]\cos\omega t + (-a_1\omega - \omega(a_1 + a_2\omega t))\sin\omega t$$
$$2y_p(t) = 2a_1 t \cos\omega t + 2a_2 t \sin\omega t$$

I. (b) continued

$$y_p''(t)+2y_p(t) = [\omega(2a_2 - a_1\omega t) + 2a_1 t]\cos\omega t + [\omega[-2a_1 - a_2\omega t] + 2a_2 t]\sin\omega t$$
$$= \epsilon \sin\omega t$$

since $\omega=\sqrt{2}$ $\quad = 2\sqrt{2}\, a_2 \cos\sqrt{2}t - 2\sqrt{2}\, a_1 \sin\sqrt{2}t = \epsilon\sin\sqrt{2}t$

$$\therefore\ 2\sqrt{2}\, a_2 = 0 \Rightarrow a_2 = 0$$
$$-2\sqrt{2}\, a_1 = \epsilon \Rightarrow a_1 = \frac{-\epsilon}{2\sqrt{2}}$$

$$\therefore\ y_p(t) = \frac{-\epsilon}{2\sqrt{2}} t\cos\sqrt{2}t$$

and

$$y(t) = \frac{-\epsilon}{2\sqrt{2}} t\cos\sqrt{2}t + c_1\cos\sqrt{2}t + c_2\sin\sqrt{2}t$$

II. $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = 0$

$$r^2+3r+2=0$$
$$(r+2)(r+1)=0$$
$$\boxed{y(t) = c_1 e^{-2t} + c_2 e^{-t}}$$

III. $\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 9y = 4t+3$

$r^2+6r+9=0 \Rightarrow (r+3)^2=0$ $\quad -3$ is a double root

$$y_h(t) = a_1 e^{-3t} + a_2 t e^{-3t}$$

By undetermined coefficients, $y_p(t) = B_0 + B_1 t$

$$y_p'(t) = B_1$$
$$y_p''(t) = 0$$

$$\frac{d^2y_p}{dt^2} + 6\frac{dy_p}{dt} + 9y_p(t) = 6B_1 + 9(B_0+B_1t) = 4t+3$$

$$\therefore\ 9B_1 = 4 \Rightarrow B_1 = 4/9$$
$$6B_1 + 9B_0 = 3 \Rightarrow 6(4/9)+9B_0 = 3$$
$$B_0 = \frac{3-\frac{8}{3}}{9} = \frac{1}{27}$$

$$y(t) = y_p(t) + y_h(t) = \frac{1}{27} + \frac{4}{9}t + a_1 e^{-3t} + a_2 t e^{-3t}$$

IV. Use annihilators to solve $\frac{d^4y}{dt^4} - \frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} = t^2$

$$L(D) = D^4 - D^3 - 6D^2 = D^2(D^2 - D - 6)$$
$$= D^2(D-3)(D+2)$$

D^3 annihilates t^2.

Consider $D^3 D^2(D-3)(D+2) = D^5(D-3)(D+2)$ annihilates

$$a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 e^{3t} + a_6 e^{-2t}$$

$D^2(D-3)(D+2)$ annihilates $a_0 + a_1 t + a_5 e^{3t} + a_6 e^{-2t}$

$\therefore$ $y_p(t) = a_2 t^2 + a_3 t^3 + a_4 t^4$

$y_p'(t) = 2a_2 t + 3a_3 t^2 + 4a_4 t^3$

$y_p''(t) = 2a_2 + 6a_3 t + 12a_4 t^2$

$y_p'''(t) = 6a_3 + 24a_4 t$

$y''''(t) = 24a_4$

$y'''' - y''' - 6y'' = 24a_4 - 6a_3 - 12a_2 + (-24a_4 - 36a_3)t - 72a_4 t^2 = t^2$

$-72a_4 = 1 \Rightarrow a_4 = \frac{-1}{72}$; $-24a_4 - 36a_3 = 0 \Rightarrow 2 - 36a_3 = 0 \Rightarrow a_3 = \frac{1}{18}$

$0 = 24a_4 - 6a_3 - 12a_2 = 24(-\frac{1}{72}) - 6(\frac{1}{18}) - 12a_2 = -2 - \frac{1}{3} - 12a_2$

$a_2 = \frac{2 + \frac{1}{3}}{-12} = \frac{\frac{7}{3}}{-12} = -7/36$

$y(t) = -7/36\, t^2 + \frac{1}{18} t^3 - \frac{1}{72} t^4 + a_0 + a_1 t + a_5 e^{3t} + a_6 e^{-2t}$

V. Find a particular solution to $y^{(3)} - 10y^{(2)} + 21y' = 1$ using variation of parameters

$$r^3 - 10r^2 + 21r = r(r^2 - 10r + 21) = r(r-3)(r-7)$$

$$y_h(t) = c_1(1) + c_2 e^{3t} + c_4 e^{7t}$$

$$W(y_1, y_2, y_3)(t) = \begin{vmatrix} 1 & e^{3t} & e^{7t} \\ 0 & 3e^{3t} & 7e^{7t} \\ 0 & 9e^{3t} & 49e^{7t} \end{vmatrix} = e^{10t} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 7 \\ 0 & 9 & 49 \end{vmatrix}$$

$$= e^{10t}(147 - 63) = 84 e^{10t}$$

$$u_1' = \frac{\begin{vmatrix} 0 & e^{3t} & e^{7t} \\ 0 & 3e^{3t} & 7e^{7t} \\ 1 & 9e^{3t} & 49e^{7t} \end{vmatrix}}{W} = \frac{e^{10t}}{84 e^{10t}} \begin{vmatrix} 0 & 1 & 1 \\ 0 & 3 & 7 \\ 1 & 9 & 49 \end{vmatrix}$$

$$= \frac{4e^{10t}}{84 e^{10t}} = \frac{1}{21}$$

$$u_1 = \frac{1}{21} t$$

$$u_2'(t) = \frac{\begin{vmatrix} 1 & 0 & e^{7t} \\ 0 & 0 & 7e^{7t} \\ 0 & 1 & 49e^{7t} \end{vmatrix}}{84 e^{10t}} = \frac{e^{7t}}{84 e^{10t}} \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 7 \\ 0 & 1 & 49 \end{vmatrix} = \frac{-7e^{7t}}{84 e^{10t}}$$

$$u_2' = -\frac{1}{12} e^{-3t} \Rightarrow u_2 = \frac{1}{36} e^{-3t}$$

$$u_3' = \frac{\begin{vmatrix} 1 & e^{3t} & 0 \\ 0 & 3e^{3t} & 0 \\ 0 & 9e^{3t} & 1 \end{vmatrix}}{84 e^{10t}} = \frac{e^{3t}}{84 e^{10t}} \begin{vmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 9 & 1 \end{vmatrix} = \frac{3}{84} e^{-7t}$$

$$= \frac{1}{28} e^{-7t}, \quad -\frac{1}{28(7)} e^{-7t} = u_3$$

$$y_p(t) = u_1 1 + u_2 e^{3t} + u_3 e^{7t}$$

$$= \boxed{\frac{1}{21}t + \frac{1}{36} - \frac{1}{28(7)}}$$