MAT 162 Final Exam Name Key 240 total

Spring 2014 version 2

For full credit, show all work.

I. Calculate the following"

10 a. $\int x^2(9+x)^{1/2}\,dx = \int (u-9)^2 u^{\frac{1}{2}}\,du$

$u = 9+x$

$u - 9 = x$

$du = dx$

$= \int u^{2.5} - 18u^{1.5} + 81u^{.5}\,du$

$= \frac{u^{3.5}}{3.5} - \frac{18u^{2.5}}{2.5} + \frac{81u^{1.5}}{1.5} + C$

$= \frac{(9+x)^{3.5}}{3.5} - \frac{18}{2.5}(9+x)^{2.5} + \frac{81}{1.5}(9+x)^{1.5} + C$

10 b. $\int x^2(9+x^2)^{1/2}\,dx = \int 3^2\tan^2\theta \; 3\sec\theta \; 3\sec\theta\tan\theta\,d\theta$

$x = 3\tan\theta$

$9+x^2 = 9\sec^2\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$

$= 3^4\int(\sec^2\theta - 1)\sec^2\theta \; \sec\theta\tan\theta\,d\theta$

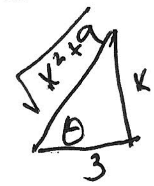
$= 3^4\int(w^4 - w^2)\,dw$

$w = \sec\theta$

$dw = \sec\theta\tan\theta\,d\theta$

$= 3^4\left(\frac{w^5}{5} - \frac{w^3}{3}\right) + C$

$= 3^4\left(\frac{1}{5}\left(\frac{x^2+9}{3}\right)^{5/2} - \frac{1}{3}\left(\frac{\sqrt{x^2+9}}{3}\right)^3\right) + C$



$\sec\theta = \frac{\sqrt{x^2+9}}{3}$

II. Tell whether $\int_1^{+\infty} \frac{2x^2 + \cos(43x)^{32}}{x^4+2x+17}\,dx$ converges or diverges, and why.

10 $\frac{2x^2 + \cos(43x)^{32}}{x^4+2x+17} \le \frac{2x^2 + x^2}{x^4} = \frac{2}{x^2}$ for $x \ge 1$

$\int_1^{+\infty} \frac{2}{x^2}\,dx$ converges $\Rightarrow \int_1^{+\infty} \frac{2x^2 + \cos(43x)^{32}}{x^4+2x+17}\,dx$ converges

$p = 2 > 1$ function

comparison test

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III. Use the trapezoidal rule with n = 5 to estimate $\int_1^5 \frac{x^2}{1+x^4}\,dx$.

$\Delta x = \frac{5-1}{5} = \frac{4}{5}$

1 1.8 2.6 3.4 4.2 5

11

10

$f(x) = \frac{x^2}{1+x^4}$

$\frac{\frac{4}{5}}{2}\left[f(1) + 2f(1.8) + 2f(2.6) + 2f(3.4) + 2f(4.2) + f(5)\right]$

$= .6711180722$

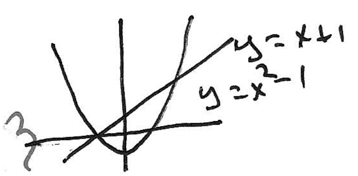
IV. Find the length of the graph of the curve $y = f(x)$, $0 \le x \le 7$, if $dy/dx = (5+3x)^{.5}$.

$L = \int_0^7 \sqrt{1+(5+3x)}\,dx = \frac{2}{3}\cdot\frac{1}{3}(6+3x)^{3/2}\Big|_0^7$

$= \frac{2}{9}\left[27^{3/2} - 6^{3/2}\right] = 27.9109$

10

V. Find the centroid of the region bounded by the curves $y = x + 1$, $y = x^2 - 1$.



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$x^2 - 1 = x + 1$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = -1$ and $x = 2$

2 $M = \int_{-1}^{2} \rho((x+1) - (x^2-1))\,dx$

$= \int_{-1}^{2} \rho(x - x^2 + 2)\,dx = \rho\left(\frac{1}{2}x^2 - \frac{1}{3}x^3 + 2x\right)\Big|_{-1}^{2}$

$= \rho\left(\frac{1}{2}2^2 - \frac{1}{3}2^3 + 2(2) - \left[\frac{1}{2} + \frac{1}{3} - 2\right]\right) = \rho 4.5$

2 $M_x = \int_{-1}^{2} \rho\frac{1}{2}\left((x+1)^2 - (x^2-1)^2\right)dx$

$= \frac{\rho}{2}\left(\frac{(x+1)^3}{3} - \left(\frac{x^5}{5} - 2\frac{x^3}{3} + x\right)\right)\Big|_{-1}^{2} = \rho\ \cancel{2.14335}\ 2.7$

2 $M_y = \int_{-1}^{2} \rho x\left((x+1) - (x^2-1)\right)dx$

$= \rho\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{2}x^2\right)\Big|_{-1}^{2} = 2.25$

2

$\bar{x} = \frac{M_y}{M}$

$\bar{y} = \frac{M_x}{M}$

2 $\bar{y} = \frac{M_x}{M} = \frac{\cancel{2.14335}\ 2.7}{4.5} = .6$

2 $\bar{x} = \frac{2.25}{4.5} = \frac{1}{2}$

35

10

VI. Find k so that $f(x) = \dfrac{k}{x^2+10x}$ if $x \geq 4$ and $f(x) = 0$ if $x < 4$, is a probability density function.

$$1 = \int_4^{+\infty} \frac{k}{x^2+10x} = \lim_{b\to+\infty} k\frac{1}{10}\ln\frac{x}{x+10}\Big|_4^b = -\frac{k}{10}\ln\frac{4}{14} = \frac{k}{10}\ln\frac{14}{4}$$

$$\boxed{k = \frac{10}{\ln(\frac{14}{4})}} \approx 7.98306$$

$$\frac{1}{x(x+10)} = \frac{A}{x} + \frac{B}{x+10} = \frac{1}{10}\frac{1}{x} - \frac{1}{10}\frac{1}{x+10}$$

$1 = A(x+10) + Bx$

$1 = A(10) \Rightarrow A = \frac{1}{10}$

$1 = B(-10) \Rightarrow B = -\frac{1}{10}$

VII. Solve completely:

(a) $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x}$, $y(0) = 2$.

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$\frac{1}{1+y^2}dy = \frac{1}{1+x}dx$ 2

$\arctan y = \ln(1+x) + C$ 3

$\arctan 2 = \ln 0 + C$ 1

$C = \arctan 2$ 2

1 $\arctan y = \ln(1+x) + \arctan 2$

1 $y = \tan[\ln(1+x) + \arctan 2$

(b) $\dfrac{dy}{dx} - 3y = 5e^{4x}$

10

$\mu(x) = e^{\int -3dx} = e^{-3x}$ 3

$\frac{d}{dx}(e^{-3x}y) = 5e^{x}$ 3

$e^{-3x}y = 5e^{x} + C$ 3

$\boxed{y = 5e^{4x} + Ce^{3x}}$ 1

(c) $\dfrac{d^2y}{dx^2} - 12\dfrac{dy}{dx} - 45y = 0.$

10

$r^2 - 12r - 45 = 0$ 2

$(r-15)(r+3)$ 2

$r = 15 \quad r = -3$

$\boxed{y(x) = c_1e^{15x} + c_2e^{-3x}}$

2 2

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VIII. Use Euler's Method and a stepsize of h = 0.1 to estimate y(.2) where $\frac{dy}{dx} = x^3+(1+y)$, y(0) = 3.

x	y	$\frac{dy}{dx}(.1)$
0	3	$(0^3+(1+3))(.1) = .4$
.1	3.4	$((.1)^3+(1+3.4)).1 = .0001+.44 = .4401$
.2	3.84	.01

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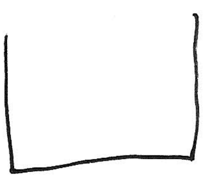
IX. A 1000 liter tank is initially filled with brine that contains 5 kg of dissolved salt. A salt solution of .004 kg/l enters the tank at a rate of 50 l/minute; the tank is continuously mixed and a solution drains from the tank at a rate of 60 l/minute. How much salt was in the tank 25 minutes later?



$S(0) = 5$ kg

$V(t) = 1000 - 10t$

rate in = $.004(50)$ kg/min

rate out = $\frac{S(t)}{1000-10t}$ 60 kg/min.

15

$\frac{ds}{dt} = .2 - \frac{6}{100-t} s(t)$

$\frac{ds}{dt} + \frac{6}{100-t} s(t) = .2$

$\mu(t) = e^{\int \frac{6}{100-t}\,dt} = e^{-6\ln(100-t)} = (100-t)^{-6}$

$\frac{d}{dt}\left((100-t)^{-6} s(t)\right) = .2(100-t)^{-6}$

$(100-t)^{-6} s(t) = \frac{.2}{5}(100-t)^{-5} + C$

$S(t) = \frac{.2}{5}(100-t) + c(100-t)^6$

$5 = S(0) = .04(100) + c(100^6)$

$1 = c(100^6)$

$C = 100^{-6}$

$S(t) = .04(100-t) + 100^{-6}(100-t)^{+6}$

$S(25) = .04(75) + 100^{-6}(75^6)$

$= .04(75) + (.75)^6 = 3.177978516$

/25

X. Find the foci and vertices and sketch the graph of $y^2 + x^2 + 16x = 24$.

$x^2 + 16x + 64 + y^2 = 24 + 64 = 88$

$(x+8)^2 + y^2 = (\sqrt{88})^2$

$$\frac{(x+8)^2}{(\sqrt{88})^2} + \frac{y^2}{(\sqrt{88})^2} = 1$$

15

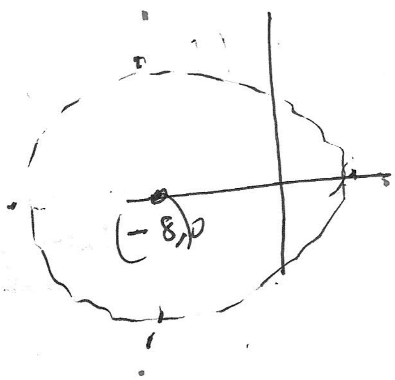
circle

center $(-8, 0)$

vertices $(-8 \pm \sqrt{88}, 0)$

$(-8, 0 \pm \sqrt{88})$

foci $(8, 0)$



XI. Convert $r = 9\sin(\theta)$ into rectangular coordinates and sketch the graph. Find the slope of the tangent line at $\theta = \frac{\pi}{2}$.

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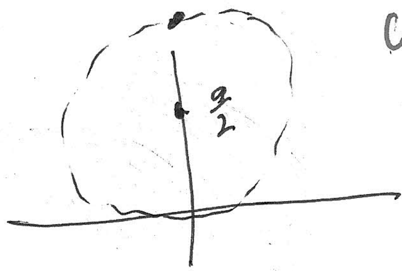
$y = r \sin\theta$

$r^2 = 9r\sin\theta$

$x^2 + y^2 = 9y$

$x^2 + y^2 - 9y + \left(\frac{9}{2}\right)^2 = \left(\frac{9}{2}\right)^2$

$x^2 + \left(y - \frac{9}{2}\right)^2 = \left(\frac{9}{2}\right)^2$



circle

center $\left(0, \frac{9}{2}\right)$

radius $\frac{9}{2}$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\cos\theta - r\sin\theta}{\frac{dr}{d\theta}\sin\theta + r\cos\theta}$$

$$= \frac{9\cos^2\theta - 9\sin^2\theta}{9\sin\theta\cos\theta + 9\sin\theta\cos\theta}$$

$$= \frac{0}{-9} = \boxed{0}$$

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XII. For $y = t^2$ and $x = t^3 - 3t$, $-2 < t < 2$

(a) Find the points where the parametric system has a vertical tangent line.

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$$\frac{dy}{dt} = 3t^2 - 3 = 3(t-1)(t+1)$$

t	x	y
-1	2	1
1	-2	1

(b) Find the points where there are horizontal tangent lines.

5

$$\frac{dy}{dt} = 2t \Rightarrow t = 0$$

t	x	y
0	0	0

(c) Find where x is increasing.

5

$t < -1$ and $t > 1$

(d) Find where y is increasing.

5

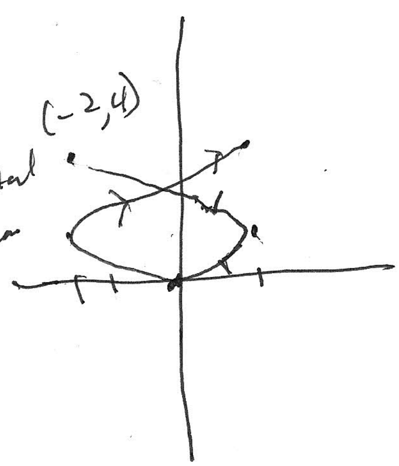
$t > 0$

(e) Sketch the graph of the system on an x-y coordinate system.

5

t	x	y
-2	2	4
-1	2	1
0	0	0
1	-2	1
2	2	4

vertical (-2,4)
horizontal
Vertical



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XIII. Tell why each series is conditionally convergent, absolutely convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{4^{n^2}}$

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} \left|\frac{(n+1)^3}{4^{(n+1)^2}} \cdot \frac{4^{n^2}}{n^3}\right|$$
$$= \lim_{n\to+\infty} \left|\left(1+\frac{1}{n}\right)^3 \cdot \frac{1}{4^{2n+1}}\right| = 0$$

10

$\therefore$ series converges absolutely

answer = 5
reason = 5

(b) $\sum_{n=1}^{\infty} (-1)^n 10^n e^{-2n}$

$$\lim_{n\to+\infty} |a_n| = \lim_{n\to+\infty} \left(\frac{10}{e^2}\right)^n \neq 0$$

10

$\therefore$ divergence.

(c) $\sum_{n=2}^{\infty} \frac{(-1)^n \ln(\ln(n))}{\ln(n)}$

n = 2

$\sum_{n=2} \frac{\ln(\ln x)}{\ln n}$ does not converge

since $\frac{\ln(\ln n)}{n \ln n} \leq \frac{\ln(\ln n)}{\ln n}$

10

and $\int_1^{+\infty} \frac{\ln(\ln x)}{x \ln x}\,dx$ does not converge (antiderivative is $\ln(\ln x) \to +\infty$ as $b \to +\infty$)

But $\sum_{n=2}^{\infty} (-1)^n \frac{\ln(\ln n)}{\ln n}$ satisfies Alt. Series test

(since $f'(x) < 0$ for $f(x) = \frac{\ln(\ln x)}{\ln}$

$f'(x) = \frac{\ln x\left(\frac{1}{x \ln x}\right) - \ln(\ln x)\frac{1}{x}}{(\ln x)^2}$

$= \frac{1}{x} \frac{(1 - \ln(\ln x))}{(\ln x)^2} < 0.$

$\therefore$ $\sum_{n=2}^{\infty} (-1) \frac{\ln(\ln n)}{\ln n}$ converges conditionally

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XIV. Find the radius and interval of convergence for $f(x) = \sum_{n=1}^{\infty}(2x+9)^n 3^{-4n}$.

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < \lim_{n\to\infty}$

$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{|2x+9|}{3^4} < 1$ 4

$-3^4 < 2x+9 < 3^4$

$-9-3^4 < 2x < -9+3^4$

$\frac{1}{2}(-9-3^4) < x < \frac{1}{2}(-9+3^4)$ 4

at $x = \frac{1}{2}(-9+3^4)$ 2

$\sum_{n=1}^{\infty} 1$ does not converge

at $x = \frac{1}{2}(-9-3^4)$

$\sum_{n=1}^{\infty}(-1)^n$ doesn't converge 2

14

radius of convergence is $\frac{3^4}{2}$ 1

$\therefore$ interval of convergence is

XV. Use a power series to estimate $\int_0^{0.1} \frac{\sin(x^5)}{4x^4}\,dx$ with an error less than 10^{-25}.

$2 \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

16

$2 \sin(x^5) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{10k+5}}{(2k+1)!}$

$2 \frac{1}{4} \frac{\sin(x^5)}{x^4} = \frac{1}{4}\sum_{k=0}^{\infty} (-1)^k \frac{x^{10k+1}}{(2k+1)!}$

$2 \int_0^{0.1} \frac{\sin(x^5)}{4x^4}\,dx = \frac{1}{4}\sum_{k=0}^{\infty} (-1)^k \frac{x^{10k+2}}{(2k)!\,(10k+2)} \Big|_0^{0.1}$

$\frac{.1}{10^{54}}$ 22

$< 10^{-25}$

$= \frac{1}{2}\sum_{k=0}^{\infty} (-1)^k \frac{.1^{10k+2}}{(2k)!\,(10k+2)}$ 3 2

$= \boxed{\frac{1}{2}\frac{(.1)^2}{2} - \frac{1}{2}\frac{(.1)^{12}}{2!\,(12)} + \frac{1}{2}\frac{(.1)^{22}}{4!\,(22)}} - \frac{1}{2}\frac{(.1)^{26}}{6!\,(26)}$

is the answer

3

0

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