240 Total

MAT 162 Final Exam Name Key

Spring 2014 version 1

−2 if either rules.

For full credit, show all work.

or $u = x^2$ $dv = (1+9x)^{1/2}$ $uv - \int v\,du = \frac{2}{27}x^2(1+9x)^{3/2}$

$du = 2x\,dx$ $v = \frac{2}{27}(1+9x)^{3/2}$

$- \int \frac{2}{27}(1+9x)^{3/2} 2x\,dx$

$= \frac{2}{27}x^2(1+9x)^{3/2} - \frac{4}{27}\int x(1+9x)^{3/2}dx$

$\int x(1+9x)^{3/2}dx = x\frac{1}{9}\frac{2}{5}(1+9x)^{5/2} - \int (1+9x)^{3/2}dx$

$= \frac{2}{45}\{x(1+9x)^{3/2} =$

I. Calculate the following”

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a. $\int x^2(1+9x)^{1/2}\,dx$

$= \int \left(\frac{u-1}{9}\right)^2 u^{\frac{1}{2}} \frac{1}{9}du$

$u = 1+9x$

$\frac{u-1}{9} = x$

$dx = \frac{1}{9}du$

$= \frac{1}{9^3}\int (u^{2.5} - 2u^{1.5} + u^{\frac{1}{2}})du$

$= \frac{1}{9^3}\left[\frac{u^{3.5}}{3.5} - \frac{2u^{2.5}}{2.5} + \frac{u^{3/2}}{1.5}\right] + C$

$\frac{1}{9^3}\left[\frac{(1+9x)^{3.5}}{3.5} - \frac{2}{2.5}(1+9x)^{2.5} + \frac{(1+9x)^{3/2}}{1.5}\right] + C$ ✓

$\frac{2(1+9x)^{3/2}(1215x^2 - 108x + 8)}{76545} =$

$\frac{2}{9^3}(1+9x)^{3/2}\left[\frac{(1+9x)^2}{7} - \frac{2}{5}(1+9x) + \frac{1}{3}\right]$ 3

$= \frac{2}{9^3}(1+9x)^{3/2}\left[\frac{15(1+9x)^2 - 42(1+9x) + 35}{7 \cdot 5 \cdot 3}\right]$

b. $\int x^3(1+9x^2)^{1/2}\,dx$

$= \int \frac{1}{3^3}\tan^3\theta \sec\theta \frac{1}{3}\sec^2\theta\,d\theta$

$\tan^2\theta$ = $\sec^2\theta - 1$

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~~$dx = \frac{1}{3}\sec^2\theta$~~

$x = \frac{\tan\theta}{3}$

$dx = \frac{1}{3}\sec^2\theta\,d\theta$

$\sqrt{1+9x^2}$

$1 + 9x^2 = 1 + \tan^2\theta$

$= \sec^2\theta$

$= \frac{1}{9^2}\int (~~1-\sec^2\theta~~)\sec^2\theta \quad \sec\theta\tan\theta\,d\theta$

$= \frac{2(1+9x^2)^{3/2}\{1215x^2 - 108x + 8\}}{76545}$

$\frac{1}{9^2}\int (~~1-w^2~~)w^2\,dw$

$= \frac{1}{9^2}\int (w^2 - 1)$

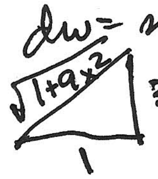
$= \frac{1}{9^2}\left(\frac{1}{3}w^3 + \frac{1}{5}w^5\right) + C$

$= \frac{1}{9^2}\left(\frac{1}{3}w^3 + \frac{1}{5}w^5\right)$

$\frac{(9x^2+1)^{3/2}(27x^2-2)}{1215}$

$w = \sec\theta$

$dw = \sec\theta\tan\theta\,d\theta$



$= \frac{1}{9^2}\left(\frac{1}{3}\sec^3\theta + \frac{1}{5}\sec^5\theta\right) + C$

$= \frac{1}{9^2}\left\{\frac{1}{3}(1+9x^2)^{3/2} + \frac{1}{5}(1+9x^2)^{5/2}\right\}$

$= \frac{1}{9^2}(1+9x^2)^{3/2}\left[\frac{1}{3} - \frac{1}{5}(1+9x^2)\right] + C$

$\frac{2 - 3 \cdot 9x^2}{5}$ 27

~~$2(1+9x)^{3/2}(\ldots)$~~

II. 15 Tell whether $\int_1^{+\infty} \frac{2x + \cos(43x)^{32}}{x^3 + 2x + 17}\,dx$ converges or diverges, and why.

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$\frac{2x + \cos(43x)^{32}}{x^3 + 2x + 17} \le \frac{2x + x}{x^3 + ~~2x~~ ~~+17~~} = \frac{3x}{x^3} = \frac{3}{x^2}$

3/2

$\int_1^{+\infty} \frac{3}{x^2}$ converges $\Rightarrow \int_1^{+\infty} \frac{2x + \cos(43x)^{32}}{x^3 + 2x + 17}$ converges.

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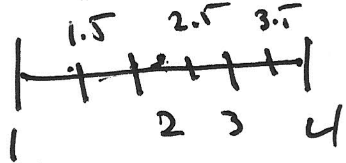
III. Use the trapezoidal rule with n = 6 to estimate $\int_1^4 \frac{x^2}{1+x^4}\,dx$.



$\Delta x = \frac{4-1}{6} = \frac{1}{2}$

$\frac{\Delta x}{2}\big(f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + 2f(3) + 2f(3.5) + f(4)\big) = .6172058\ldots$

where $f(x) = \frac{x^2}{1+x^4}$

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IV. Find the length of the graph of the curve $y = f(x)$, $0 \le x \le 3$, if dy/dx = $(5+3x)^{.5}$.

$L = \int_0^3 \sqrt{1 + \cancel{5}+3x)}\ dy$

$= \int_0^3 \sqrt{6 + 3x}\ dy$

$= \frac{2}{9}(6+3x)^{3/2}\Big|_0^3 = \frac{2}{9}\left(15^{3/2} - 6^{3/2}\right) = 9.643958164$

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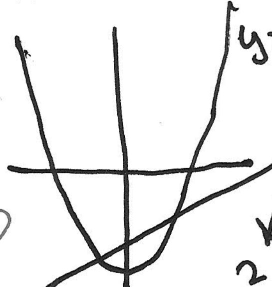
V. Find the centroid of the region bounded by the curves y = x - 10 , y = x² - 12 .



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$x^2-12 = x - 10$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

2 $x=2 \quad x=-1$

2 $M = \int_{-1}^{2} P\big((x-10) - (x^2-12)\big)\,dy$ $= 4.5P$

2 $M_y = \int_{-1}^{2} Px\big((x-10) - (x^2-12)\big)\,dy$ $= 2.25P$

2 $M_x = \int_{-1}^{2} P\,\frac{(x-10)+(x^2-12)}{2}\big((x-10)-(x^2-12)\big)\,dy$ $= -46.8P$

2 $\bar{x} = \frac{M_y}{M} = \frac{1}{2}$

2 $\bar{y} = \frac{M_x}{M} = -10.4$

15

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VI. Find k so that f(x) = $\frac{k}{x^2+6x}$ if x≥4 and f(x) = 0 if x <4, is a probability density function.

(10) $1 = \lim_{b\to+\infty}\int_4^b \frac{k}{x^2+6x} = \lim_{b\to+\infty} \frac{k}{6}\ln\frac{x}{x+6}\Big|_4^b$

$= -\frac{k}{6}\ln\frac{4}{4+6}$

$\frac{1}{x(x+6)} = \frac{A}{x} + \frac{B}{x+6} = \frac{1}{6}\left[\frac{1}{x} - \frac{1}{x+6}\right]$

$1 = A(x+6) + Bx$

$1 = A6 \Rightarrow A = \frac{1}{6}$

$B = -\frac{1}{6}$

$= \frac{k}{6}\ln\frac{10}{4} = \frac{k}{6}\ln 2.5$

$\boxed{k = \frac{6}{\ln 2.5}}$

$= 6.54814$

VII. Solve completely:

(a) $\frac{dy}{dx} = \frac{1+y}{1+x^2}$, y(0) = 2.

(10)

$\frac{1}{1+y}dy = \frac{1}{1+x^2}dx$ 2

$\ln(1+y) = \arctan x + C$ 3

$\ln 3 = 0 + C$ 2

$\ln(1+y) = \arctan x + \ln 3$ 1

$1+y = 3e^{\arctan x}$ 1

$\boxed{y = -1 + 3e^{\arctan x}}$ 1

(b) $\frac{dy}{dx}$ - 7y = 5e^{4x}

(10)

$\mu(x) = e^{-\int 7dx} = e^{-7x}$ 3

$\frac{d}{dx}(e^{-7x}y) = 5e^{-3x}$ 3

$e^{-7x}y = -\frac{5}{3}e^{-3x} + C$ 3

$\boxed{y = -\frac{5}{3}e^{4x} + Ce^{7x}}$ 1

(c) $\frac{d^2y}{dx^2} + 12\frac{dy}{dx}$ -45y = 0.

$\boxed{y(x) = c_1 e^{-15x} + c_2 e^{3x}}$

2 2

(10)

$r^2 + 12r - 45 = 0$ 2

$(r+15)(r-3) = 0$ 2

$r = -15,\ r = 3$ 2

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VIII. Use Euler's Method and a stepsize of h = 0.1 to estimate y(.2) where $\frac{dy}{dx} = x^4+(1+y)$, y(0) = 3.

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x	y	f(x,y) .1
0	3	4(.1) = .4
.1	3.4 2	[.0001 + (4.4)].1 = .44 + .00001 = .44001
.2		

$\boxed{3.84001}$ 2

IX. A 1000 liter tank is initially filled with brine that contains dissolved salt. A salt solution of .004 kg/l enters the tank at a rate of 50 l/minute; the tank is continuously mixed and a solution drains from the tank at a rate of 60 l/minute. In 50 minutes there is exactly 2 kg of salt in the tank. How much salt was in the tank in the beginning?

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S(50) = 2

S(t) = amount of salt at t minutes

$$\frac{ds}{dt} = \text{rate in} - \text{rate out}$$

$$= (.004)(50) - \frac{S(t)}{1000-10t}(60)$$

$$\frac{ds}{dt} + \frac{6}{100-t} s = .2$$

$$\mu(t) = e^{\int \frac{6}{100-t}} = e^{-6\ln(100-t)} = (100-t)^{-6}$$

$$\frac{d}{dt}\left((100-t)^{-6} s\right) = .2(100-t)^{-6}$$

$$(100-t)^{-6} s = \; = \frac{.2}{5}(100-t)^{-5} + c$$

$$s(t) = \frac{1}{25}(100-t) + c(100-t)^6$$

$$2 = s(50) = \frac{1}{25}(50) + c(50)^6$$

$$c = 0$$

$$S(t) = \frac{1}{25}(100-t)$$

$$S(0) = \frac{1}{25}(100-0) = \boxed{4 \text{ kg}}$$

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X. Find the foci and vertices and sketch the graph of $-y^2 + x^2 + 16x = 36$.

$x^2+16x+64-y^2 = 36+64=100$

$\frac{(x+8)^2}{10^2} - \frac{y^2}{10^2} = 1$

Center $(-8,0)$

vertices $(-8 \pm 10, 0) = \left\{\begin{matrix} -18, 0) \\ 2, 0 \end{matrix}\right.$

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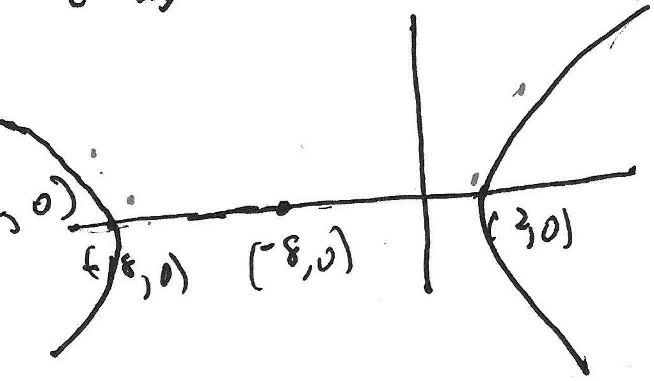
$c^2 = a^2 + b^2 = 200$

$c = 10\sqrt{2}$

~~vertices~~ $(-8 \pm 10\sqrt{2}, 0)$

foci



XI. Convert r = 9cos(θ) into rectangular coordinates and sketch the graph. Find the slope of the tangent line at $\theta = \frac{\pi}{2}$.

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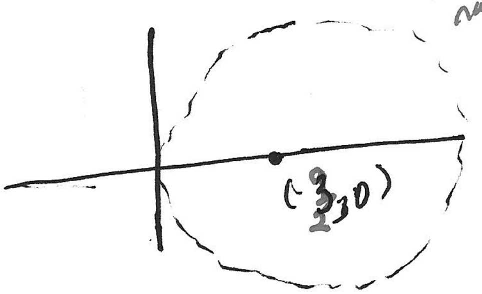
$r^2 = 9r\cos\theta$

$x^2+y^2 = 9x$ ~~$=0$~~

$x^2-9x+\left(\frac{9}{2}\right)^2+y^2=\left(\frac{9}{2}\right)^2$

$\left(x-\frac{9}{2}\right)^2+y^2=\frac{9}{2}^2$ circle center $(\frac{9}{2},0)$ radius $\frac{9}{2}$

slope is undefined at $\theta = \frac{\pi}{2}$.



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XII. For x =t^2 and y = t^3 - 3t, -2< t < 2

(a) Find the points where the parametric system has a vertical tangent line.

$x = t^2$

5 $\frac{dx}{dt} = 2t = 0$

$t = 0$

$\boxed{\begin{matrix} x = 0 \\ y = 0 \end{matrix}}$ vertical tangent line

(b) Find the points where there are horizontal tangent lines.

5 $\frac{dy}{dt} = 3t^2 - 3 = 3(t-1)(t+1)$

$t = -1, 1$

t	x	y
-1	1	2
1	1	-2

horizontal tangent lines

(c) Find where x is increasing.

5 $\frac{dx}{dt} > 0$ when $t > 0$

(d) Find where y is increasing.

5 $\frac{dy}{dt} > 0$ when $t < -1$ and $t > 1$

(e) Sketch the graph of the system on an x-y coordinate system.

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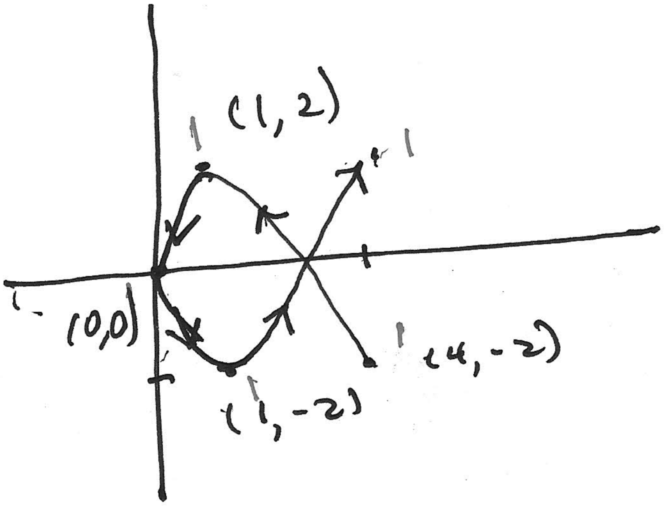
t	x	y
-2	4	-2
-1	1	2
0	0	0
1	1	-2
2	4	2



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XIII. Tell why each series is conditionally convergent, absolutely convergent or divergent.

(a) $\sum_{n=1}^{\infty}\frac{(-1)^n n^3}{e^{n^2}}$

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty}\left|\frac{(n+1)^3}{e^{(n+1)^2}}\cdot\frac{e^{n^2}}{n^3}\right|$$

$$= \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^3\frac{1}{e^{2n+1}} = 0$$

10

∴ absolutely convergent

answer 5
reason 5

(b) $\sum_{n=1}^{\infty}(-1)^n 9^n e^{-2^n}$

$$\lim_{n\to+\infty}\frac{9^n}{(e^2)^n} \neq 0$$

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∴ divergent

(c) $\sum_{n=1}^{\infty}\frac{(-1)^n \ln(n)}{n}$

$$f(x) = \frac{\ln x}{x}$$

$$f'(x) = \frac{x\left(\frac{1}{x}\right) - \ln x(1)}{x^2} = \frac{1-\ln x}{x^2} < 0 \text{ when } x > e$$

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$$\lim_{x\to+\infty} f(x) = 0$$

By ~~Alt~~ Series test $\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n}$ converges.

By $\sum_{n=1}\frac{\ln n}{n}$ diverges $\frac{\ln n}{n} > \frac{1}{n}$ for $n \geq e$

∴ Series is conditionally convergent

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XIV. Find the radius and interval of convergence for $f(x) = \sum_{n=1}^{\infty}(3x+9)^n 2^{-4n}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3x+9)^{n+1}}{2^{4(n+1)}} \cdot \frac{2^{4n}}{(3x+9)^n}\right| = \frac{|3x+9|}{2^4} < 1 \quad 4$$

⑭

$$\therefore |x+3| < \frac{1}{3}2^4$$

$$\boxed{R = \frac{1}{3}(2^4)}. \quad -\frac{1}{3}2^4 < x+3 < \frac{1}{3}2^4$$

$$-3-\frac{1}{3}2^4 < x < -3+\frac{1}{3}2^4 \quad 4$$

at $x = -3+\frac{1}{3}2^4$, $3(x+3) = 2^4$ $\quad \sum_{n=1}^{\infty} 1$ diverges 2

at $x = -3-\frac{1}{3}2^4$ $\quad 3(x+3) = -2^4$ $\quad \sum_{n=1}^{\infty}(-1)^n 1$ diverges 2

$\therefore$ interval of convergence is ——

XV. Use a power series to estimate $\int_0^{0.1} \frac{\sin(x^5)}{4x^2}\,dx$ with an error less than 10^{-25}.

$$2 \quad \sin x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$2 \quad \sin(x^5) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{10k+5}}{(2k+1)!}$$

⑯

$$2 \quad \therefore \frac{1}{4x^2}\sin(x^5) = \frac{1}{4}\sum_{k=0}^{\infty}(-1)^k \frac{x^{10k+3}}{(2k+1)!}$$

$$2 \quad \int \frac{1}{4x^2}\sin(x^5) = C + \frac{1}{4}\sum_{k=0}^{\infty}(-1)^k \frac{x^{10k+4}}{(2k+1)!\,(10k+4)}$$

$$2 \quad \int_0^{0.1} \frac{\sin(x^5)}{4x^2}\,dx = \frac{1}{4}\sum_{k=0}^{\infty}(-1)^k \frac{.1^{10k+4}}{(2k+1)!\,(10k+4)} \quad 3 \quad < 10^{-25}$$

$$= \boxed{\frac{1}{4}\,\frac{1}{4} - \frac{1}{4}\,\frac{(.1)^{14}}{5!\,(14)}} + \frac{1}{4}\,\frac{(.1)^{24}}{5!\,(24)}$$

is the ~~[illegible]~~ ans. 3

$\therefore$

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