Solve the following:

Hint: ~~one~~ one is 1st order linear another separable, homogeneous, Bernoulli, exact, exact after integrating factor.

1. $\dfrac{dy}{dx} = \dfrac{2xy}{x^2-y^2}$

2. $\dfrac{dy}{dx} = \dfrac{4-2x}{3y^2-5}$

3. $(x^2+1)\dfrac{dy}{dx} + 3xy = 6x$

4. $x\dfrac{dy}{dx} + 6y = 3xy^{4/3}$

5. $\dfrac{dy}{dx} = \dfrac{y^3 - 6xy}{4y + 3x^2 - 3y^2}$

6. $\dfrac{dy}{dx} = \dfrac{-y^2\cos x}{4 + 5y\sin x}$

7. A community contains 15000 people who are susceptable to a spreading contagious disease. At time $t=0$, the number $N(t)$ of people who have the disease is 5000 and is increasing by 500 per day. How long will it take for another 5000 people to contract the disease? Assume that $N'(t)$ is proportional to the product of the numbers of those who have the disease and those who do not.

8. For $\dfrac{dy}{dt} = t^3 + y^3$, $y(0) = 1$.
Estimate $y(.2)$ using a ~~stepsize~~ stepsize of $h = .1$

9. Sketch the solutions to $\dfrac{dy}{dx} = (y-2)(y-4)^2(y-6)$
for $y(0) = 1$, $y(0) = 3$, $y(0) = 5$, $y(0) = 8$.

(1) $\frac{dy}{dx} = \frac{2xy}{x^2-y^2} = \frac{2\frac{y}{x}}{1-\left(\frac{y}{x}\right)^2}$ $\quad v = \frac{y}{x} \quad y = vx$

$$\frac{dv}{dx}x + v = \frac{2v}{1-v^2}$$

$$\frac{dv}{dx}x = \frac{2v}{1-v^2} - \frac{v(1-v^2)}{1-v^2} = \frac{v(2-1+v^2)}{1-v^2} = -\frac{v(v^2+1)}{v^2-1}$$

$$\int \frac{v^2-1}{v(v^2+1)}\,dv = \int -\frac{1}{x}\,dx$$

$$\int \left(-\frac{1}{v} + \frac{2v}{v^2+1}\right)dv = \int -\frac{1}{x}\,dx$$

$$\ln\frac{v^2+1}{v} = -\ln x + c$$

$$\ln\frac{x(v^2+1)}{v} = c$$

$$\frac{x(v^2+1)}{v} = k \Rightarrow x(v^2+1) = kv$$

$$x\left(\frac{y^2}{x^2}+1\right) = k\frac{y}{x}$$

$$\boxed{y^2 + x^2 = ky}$$

$$y^2 - ky + x^2 = 0$$

$$y^2 - ky + \frac{k^2}{4} + x^2 = \frac{k^2}{4}$$

$$\left(y-\frac{k}{2}\right)^2 + x^2 = \frac{k^2}{4}$$

(2) $\frac{dy}{dx} = \frac{4-2x}{3y^2-5}$

$$(3y^2-5)\,dy = (4-2x)\,dx$$

$$\boxed{y^3 - 5y = 4x - x^2 + C}$$

(3) $(x^2+1)\frac{dy}{dx} + 3xy = 6x$

$$\frac{dy}{dx} + \frac{3}{2}\frac{2x}{x^2+1}y = 6\frac{x}{x^2+1}$$

$$\mu = e^{\frac{3}{2}\ln(x^2+1)} = (x^2+1)^{3/2}$$

$$\frac{d}{dx}\left(y(x^2+1)^{3/2}\right) = 6x(x^2+1)^{\frac{1}{2}}$$

$$y(x^2+1)^{3/2} = 2(x^2+1)^{3/2} + C$$

$$y = 2 + c(x^2+1)^{-3/2}$$

4. $x\frac{dy}{dx} + 6y = 3xy^{4/3}$ This is a Bernoulli Equation

Let $v = y^{1-4/3} = y^{-\frac{1}{3}}$

$\frac{dv}{dx} = -\frac{1}{3}y^{-4/3}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -3y^{4/3}\frac{dv}{dx}$

$x(-3y^{4/3})\frac{dv}{dx} + 6y = 3xy^{4/3}$

$\frac{dv}{dx} - \frac{2}{x}y^{-\frac{1}{3}} = -1$ first order linear

$\mu = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = x^{-2}$

$\frac{d}{dx}(x^{-2}v) = -\frac{1}{x^2}$

$x^{-2}v = -\frac{1}{x^2}$

$x^{-2}v = \frac{1}{x} + C$

$v = x + Cx^2 \Rightarrow y^{-\frac{1}{3}} = x + Cx^2$

$\boxed{y = \frac{1}{(x+Cx^2)^3}}$ Answer

check

$y(x+Cx^2)^3 = 1$

$\frac{dy}{dx}(x+Cx^2)^3 + 3y(x+Cx^2)^2(1+2Cx) = 0$

Multiply by $\frac{x}{(x+Cx^2)^3}$:

$x\frac{dy}{dx} + \frac{3yx}{(x+Cx^2)}(1+2Cx) = 0$

$x\frac{dy}{dx} + \frac{3y(2x+2Cx^2)}{(x+Cx^2)} = 0$

$x\frac{dy}{dx} + 6y\frac{(x+Cx^2)}{(x+Cx^2)} - \frac{3yx}{x+Cx^2} = 0$

$x\frac{dy}{dx} + 6y = \frac{3xy}{x+Cx^2} = 3xyy^{1/3} = 3xy^{4/3}$.

Note: 4x was in the box inside of 4y for the problem given to you with 4x the problem is not exact; as we discussed the problem various alterations to the problem were given to make it exact. I will grade your work based upon the problem you attempted.

5. $\frac{dy}{dx} = \frac{y^3 - 6xy}{\boxed{4y} + 3x^2 - 3xy^2}$

$(6xy - y^3) + (4y + 3x^2 - 3xy^2)\frac{dy}{dx} = 0$

$M = 6xy - y^2$; $N = 4y + 3x^2 - 3xy^2$

$\frac{\partial M}{\partial y} = 6x - 3y^2 \equiv \frac{\partial N}{\partial x}$.

this ~~exqu~~ equation is exact.

$\therefore$ Using $\frac{\partial F}{\partial x} = M$

$F(x,y) = \int (6xy - y^2)\,dx = 3x^2y - xy^3 + g(y)$

$N = \frac{\partial F}{\partial y}$

$4y + \cancel{3x^2} - \cancel{3xy^2} = \cancel{3x^2} - \cancel{3xy^2} + g'(y)$

$4y = g'(y) \Rightarrow g(y) = 2y^2$

$\therefore$ the general solution is $\boxed{3x^2y - xy^3 + 2y^2 = c}$

---

6. $\frac{dy}{dx} = \frac{-y^2\cos x}{4 + 5y\sin x}$

$\underbrace{y^2\cos x}_{M} + \underbrace{(4 + 5y\sin x)}_{N}\frac{dy}{dx} = 0$

$M_y = 2y\cos x \neq N_x = 5y\cos x$

but $\frac{N_x - M_y}{M} = \frac{5y\cos x - 2y\cos x}{y^2\cos x} = \frac{3}{y}$ is a function of y

$\mu \ni \frac{d\ln\mu}{dy} = \frac{3}{y}$

$\ln\mu = 3\ln y$

$\mu = y^3$

$\therefore y^5\cos x + [4y^3 + 5y^4\sin x]\frac{dy}{dx} = 0$ is exact.

$\therefore F(x,y) = \int y^5\cos x\,dx = y^5\sin x + g(y)$

$5y^4\cancel{\sin x} + g'(y) = 4y^3 + 5y^4\cancel{\sin x} \Rightarrow g'(y) = 4y^3 \Rightarrow g(y) = y^4$

$\boxed{y^5\sin x + y^4 = c}$

#7 $N(0) = 5000$
$N'(0) = 500$

$N'(t) = k\, N(t)(15000 - N(t))$

$500 = N'(0) = k\, N(0)\,(15000 - N(0)) = k(5000)(10000)$

$k = \dfrac{500}{(5000)(10000)} = 10^{-5}$

$N'(t) = 10^{-5} N(t)\,(15000 - N(t))$

$\dfrac{1}{N(15000-N)}\, dN = 10^{-5}\, dt$

$\dfrac{1}{N(15000-N)} = \dfrac{A}{N} + \dfrac{B}{15000-N} \Rightarrow 1 = A(15000-N)$

$1 = BN$

Let $N=0$ $\quad 1 = A(15000) \Rightarrow A = \dfrac{1}{15000}$

$N = 15000 \quad 1 = B\,15000 \quad \Rightarrow B = \dfrac{1}{15000}$

$\displaystyle\int \frac{1}{15000}\left(\frac{1}{N} + \frac{1}{15000-N}\right) dN = \int 10^{-5}\, dt$

$\dfrac{1}{15000} \ln N - \ln(15000 - N) = 10^{-5} t + C$

At $t=0$,

$\dfrac{1}{15000}(\ln 5000 - \ln(10000)) = 10^{-5}(0) + C$

$\dfrac{1}{15000} \ln \dfrac{1}{2} = C$

$\dfrac{1}{15000}\left[\ln N - \ln(15000 - N)\right] = 10^{-5} t + \dfrac{1}{15000} \ln \dfrac{1}{2}$

Solve for t when $N = 10,000$

$\dfrac{1}{15000}\left[\ln 10,000 - \ln 5000\right] = 10^{-5} t + \dfrac{1}{15000} \ln \dfrac{1}{2}$

$\dfrac{1}{15000}\left[\ln 2 - \ln \dfrac{1}{2}\right] = 10^{-5} t$

$t = \dfrac{10^5}{15000} \ln 4 = \dfrac{10}{1.5} \ln 4$ days $= \boxed{9.24 \text{ days}}$

8.

t	y	$f(t,y)h$
0	1	.1
.1	1.1	.1332
1.1	$\boxed{1.2332}$	

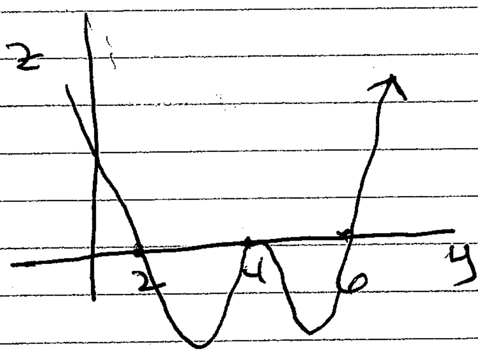
$f(t,y) = t^3 + y^3$

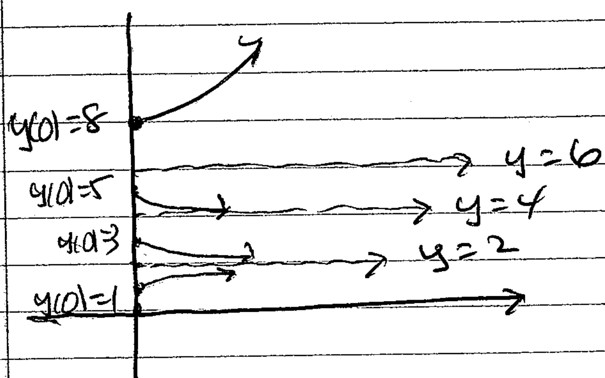
$f(0,1)\cdot h = (0^3+1^3)(.1) = .1$

$f(.1,1.1)\cdot h = ((.1)^3+(1.1)^3)(.1)$

$= .1332$

9. $z = g(y) = (y-2)(y-4)^2(y-6)$





Second Order Linear Equations with Constant Coefficients

$(*)\ a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + c\,y(t) = 0$, $y(t_0) = y_0$, $y'(t_0) = y_1$

Assume $y(t) = e^{rt}$
$y'(t) = re^{rt}$
$y''(t) = r^2 e^{rt}$

$(**)\ ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$
$ar^2 + br + c = 0$
characteristic equation
$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\boxed{\underline{\text{Case I}}\quad b^2-4ac > 0}$ $r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$
$r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$

$y_1(t) = e^{r_1 t}$ $y_2(t) = e^{r_2 t}$

Gen Soln to $(*)$ $\underline{y(t) = c_1e^{r_1t} + c_2e^{r_2t}}$
$y' = c_1r_1e^{r_1t} + c_2r_2e^{r_2t}$
$y'' = c_1r_1^2e^{r_1t} + c_2r_2^2e^{r_2t}$

$ay'' + by' + cy = c_1e^{r_1t}\underbrace{(ar_1^2+br_1+c)}_{0} + c_2e^{r_2t}\underbrace{(ar_2^2+br_2+c)}_{0}$
$= 0$

$$\boxed{\begin{aligned} y_0 = y(t_0) &= \underline{c_1}\,e^{r_1t_0} + \underline{c_2}\,e^{r_2t_0} \\ y_1 = y'(t_0) &= \underline{c_1}\,r_1e^{r_1t_0} + \underline{c_2}\,r_2e^{r_2t_0} \end{aligned}}$$

two linear equations
in c_1 & c_2

Page 144

(12) $y'' + 3y' = 0, \quad y(0) = -2$
$y'(0) = 3$

$r^2 + 3r = 0$
$r(r+3) = 0$
$r = 0 \ \& \ r = -3$
$y(t) = c_1 e^{0 \cdot t} + c_2 e^{-3t}$
$= c_1 + c_2 e^{-3t}$
$-2 = y(0) = c_1 + c_2$
$y'(t) = c_2(-3)e^{-3t}$
$3 = y'(0) = -3c_2 e^{-3(0)}$
$c_2 = -1$
$-2 = c_1 + (-1)$
$c_1 = -1$

$\boxed{y(t) = -1 - e^{-3t}}$

Case II : $b^2-4ac=0$

$ar^2+br+c=0$ has one root

$$r=-\frac{b}{2a}$$

As before $y_1(t)=e^{rt}$ is a solution to $ay''+by'+cy=0$

Consider $g(t)=te^{rt}$

$$g'(t)=e^{rt}+tre^{rt}$$
$$=e^{rt}(1+tr)$$
$$g''(t)=e^{rt}(r+tr^2+0+r)$$
$$=e^{rt}(tr^2+2r)$$

$$ag''+bg'+cg=e^{rt}(ct)$$
$$+e^{rt}(b+btr)$$
$$+e^{rt}(2ra+atr^2)$$

$t(ar^2+br+c)=t\cdot 0=0$

$2ra+b+\cancel{\ } = 2a\left(\frac{-b}{2a}\right)+b+\cancel{\ }=0$

e^{rt} is a soln & te^{rt} is a soln if $b^2-4ac=0$ & $r=\frac{-b}{2a}$

$$\boxed{\text{gen soln } y(t)=c_1e^{rt}+c_2te^{rt}}$$

Case III $b^2-4ac<0$

$ar^2+br+c=0$ has complex roots.

$r=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ negative

Page 163

6

9

11

21

$r_1=\dfrac{-b+\sqrt{4ac-b^2}\,i}{2a}$

$r_2=\dfrac{-b-\sqrt{4ac-b^2}\,i}{2a}$

$\mathrm{Re}\,r_1+\mathrm{Re}\,r_2=\dfrac{-b}{2a}$

$\mathrm{Im}\,r_1+\mathrm{Im}\,r_2=\pm\dfrac{\sqrt{4ac-b^2}}{2a}$

$\alpha=-\dfrac{b}{2a}\qquad \beta=\dfrac{\sqrt{4ac-b^2}}{2a}$

Then $y_1(t)=e^{\alpha t}\cos\beta t$ is a soln to $ay''+by'+cy=0$

and $y_2(t)=e^{\alpha t}\sin\beta t$ is a soln to
(this takes some calculation)

gen soln is
$y(t)=c_1e^{\alpha t}\cos\beta t+c_2e^{\alpha t}\sin\beta t$

20 $y''+y=0\quad y(\frac{\pi}{3})=2,\ y'(\frac{\pi}{3})=-4$

Page 163 $r^2+1=0$

$r^2=-1$

$r=\pm i\quad \alpha=0,\ \beta=1$

$y(t)=c_1\cos t+c_2\sin t$

$2=y(\frac{\pi}{3})=c_1\frac{1}{2}+c_2\frac{\sqrt3}{2}$

$y'(t)=-c_1\sin t+c_2\cos t$

$-4=-c_1\frac{\sqrt3}{2}+c_2(\frac{1}{2})$

$2=\dfrac{c_1}{2}+\dfrac{c_2\sqrt3}{2}$

$\boxed{\begin{aligned}4&=c_1+c_2\sqrt3\\ -8&=-c_1\sqrt3+c_2\end{aligned}}$ e_2

e_1 $4\sqrt3=c_1\sqrt3+c_2 3$

$4\sqrt3-8=4c_2$

$\boxed{c_2=\sqrt3-2}$