first order linear
separable
homogeneous
models

Due Friday, August 31

Section 2.1
#9, #16

Section 2.2
#3, ~~#4~~, #32

Section 2.3
#5

Due Wednesday

Section 2.4
#30

Page 77
#27 $n=0, n=1$ linear $\quad y' + p(t)y = g(t)y^n$

$n \neq 1 \quad v = y^{1-n}$

$$\frac{dv}{dt} = (1-n)\, y^{-n} \frac{dy}{dt} \Rightarrow \frac{dy}{dt} = \frac{1}{1-n} y^n \frac{dv}{dt}$$

$$\therefore \frac{y^n}{1-n} \frac{dv}{dt} + p(t)y = g(t)y^n$$

$$v' + (1-n)p(t)\, y^{1-n} = (1-n)\, g(t)$$

$$v' + (1-n)\, p(t)\, v = (1-n)\, g(t) \quad \text{linear in } v.$$

Suppose $f(x,y) = c$ and $y = y(x)$

$$\frac{\partial f}{\partial x}(x,y) + \frac{\partial f}{\partial y}(x,y)\frac{dy}{dx} = 0$$

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

For example $x^2y^3 = c$ $(**)$

$(*)$ $2xy^3 + 3x^2y^2\frac{dy}{dx} = 0$

$\therefore$ solutions to $*$ look like $(**)$

In reverse, start with $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ (1)

If there happens to exist a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = M \quad \text{and} \quad \frac{\partial f}{\partial y} = N$$

then (1) can be written as

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{d}{dx}\left[f(x,y)\right] = 0$$

$$\therefore f(x,y) = c$$

In this case (1) is called an exact differential equations.

For example,

page 99 #1 $2x+3 + (2y-2)\frac{dy}{dx} = 0$ $(*)$

by inspection $f(x,y) = x^2 + 3x + y^2 - 2y$

has the property that $\frac{\partial f}{\partial x} = 2x+3$

$\frac{\partial f}{\partial y} = 2y - 2$

$\therefore$ solutions to $(*)$ are of the form

$$x^2 + 3x + y^2 - 2y = c$$

Suppose F_x and F_y exists for $F(x,y)$ and suppose

(1) $F(x, y(x)) = C$

then $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$

(2)

and $y(x)$ satisfies the D.E. $M(x,y) + N(x,y)\frac{dy}{dx} = 0$

where $M(x,y) = \frac{\partial F}{\partial x}$ and $N(x,y) = \frac{\partial F}{\partial y}$.

$\therefore$ if $\exists$ $F(x,y)$ $\ni$ $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$

then $F(x,y) = C$ implicitly defines a general solution of (2).

In this case (2) is called an <u>exact differential equation</u>

<u>Two Questions</u>

Q1: How can we determine if (2) is exact

<u>Q2</u>: If it exact, how can we find $F(x,y)$ so that

$$\frac{\partial F}{\partial x} = M(x,y) \text{ and } \frac{\partial F}{\partial y} = N(x,y)$$

<u>Q1</u>: Suppose F_{xy} and F_{yx} are cont in an open set of $\mathbb{R}^2$

then $F_{xy} = F_{yx}$.

$\therefore$ If (2) is exact $\frac{\partial M}{\partial y} = F_{xy} = F_{yx} = \frac{\partial N}{\partial x}$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}.$$

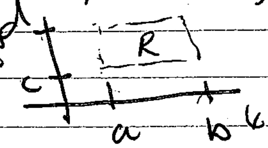
Thus a necessary condition for exactness is

(3) $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

(i.e. If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then (2) is not exact

We have done $\Rightarrow$ of the following theorem:

Thm: Suppose $M(x,y), N(x,y), \frac{\partial M}{\partial x}, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}, \frac{\partial N}{\partial y}$ are continuous in



Then $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ (2)
is ~~and~~ exact in R

$\Longleftrightarrow$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (3)$$

at each point in R.

[i.e. $\exists$ $F(x,y)$ defined in R & $\frac{\partial F}{\partial x} = M, \frac{\partial F}{\partial y} = N$ iff (3) holds]

Proof ($\Rightarrow$) done

($\Leftarrow$) Note that

(4) $$F(x,y) = \int M(x,y)\,dx + g(y)$$

satisfies $\frac{\partial F}{\partial x} = M$

We will choose $g(y)$ so that

$$N = \frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\int M(x,y)\,dx\right) + g'(y)$$

so that $g'(y) = \underbrace{N - \frac{\partial}{\partial y}\int M(x,y)\,dx}$, this is a function of y also since

$$\frac{\partial}{\partial x}\left(N - \frac{\partial}{\partial y}\int M(x,y)\,dx\right) = \frac{\partial N}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial y}\int M(x,y)\,dx$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\frac{\partial}{\partial x}\int M(x,y)\,dx$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}M = 0$$

$$\therefore F(x,y) = \int M(x,y)\,dx + \int g'(y)\,dy$$

$$= \int M(x,y)\,dx + \int\left(N - \frac{\partial}{\partial y}\int M(x,y)\,dx\right)dy$$

and $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,dx + N - \frac{\partial}{\partial y}\int M(x,y)\,dx = N.$

Q.E.D.

Example (from Edwards & Penny)

Solve
$$(6xy - y^3) + (4y + 3x^2 - 3xy^2)\frac{dy}{dx} = 0$$

Solution: $M = 6xy - y^3$ $\quad N = 4y + 3x^2 - 3xy^2$

$$\frac{\partial M}{\partial y} = 6x - 3y^2 \qquad \frac{\partial N}{\partial x} = 6x - 3y^2 \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ ~~exact~~ equation is exact.

Since $\frac{\partial F}{\partial x} = M$, $\quad F(x,y) = \int (6xy - y^3)\,dx = 3x^2y - xy^3 + g(y)$

$$\frac{\partial F}{\partial y} = \cancel{3x^2} - \cancel{3xy^2} + g'(y) = N = 4y + \cancel{3x^2} - \cancel{3xy^2}$$

$$g'(y) = 4y$$
$$g(y) = 2y^2 + C_1$$

$$F(x,y) = 3x^2y - xy^3 + 2y^2 + C_1$$

$$\therefore 3x^2y - xy^3 + 2y^2 = C$$

Alternate Solution

Since $\frac{\partial F}{\partial y} = N = 4y + 3x^2 - 3xy^2$

$$F(x,y) = \int (4y + 3x^2 - 3xy^2)\,dy = 2y^2 + 3x^2y - xy^3 + h(x)$$

$$\frac{\partial F}{\partial x} = 6xy - y^3 + h'(x) = M = 6xy - y^3$$

$$h'(x) = 0$$
$$h(x) = C$$

$$\therefore \boxed{F(x,y) = 2y^2 + 3x^2y - xy^3 = C}$$

Another version of integrating factors

From previous page: $(6xy - y^3) + (4y + 3x^2 - 3xy^2)\frac{dy}{dx} = 0$

is exact

but

$$(6x - y^2) + \left(4 + \frac{3x^2}{y} - 3xy\right)\frac{dy}{dx} = 0$$

[mult. first eq. by $\frac{1}{y}$]

is not since

$$M = 6x - y^2 \qquad N = 4 + \frac{3x^2}{y} - 3xy$$

$$M_y = -2y \neq N_x = \frac{6x}{y} - 3y$$

Assume $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ is not exact but

$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)\frac{dy}{dx} = 0$ is.

$$(\mu M)_y = (\mu N)_x$$

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

$$\mu_y M - \mu_x N + \mu(M_y - N_x) = 0$$

Assume $\mu_x = 0$.

then $\mu_y M + \mu(M_y - N_x) = 0$

$$\frac{\mu_y}{\mu} M + M_y - N_x = 0$$

$$\frac{\mu_y}{\mu} = \frac{N_x - M_y}{M}$$

$$\frac{d \ln \mu}{dy} = \frac{N_x - M_y}{M} \quad \text{is a function of } y \text{ alone}$$

[in example above $\frac{d \ln \mu}{dy} = \frac{\frac{6x}{y} - 3y}{6x - y^2} = \frac{1}{y}$

$$\ln \mu = \ln y$$

$$\mu = y \text{]}$$

Or Assume $\mu_y = 0 \Rightarrow -\mu_x N + \mu(M_y - N_x) = 0$

$$\mu(M_y - N_x) = \mu_x N$$

$$\frac{d \ln \mu}{dx} = \frac{\mu_x}{\mu} = \frac{M_y - N_x}{N} \quad \text{is a function of } x \text{ alone.}$$

To summarize : (*) $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ is exact iff $M_y = N_x$.

(*) is equivalent to $M(x,y)\frac{dx}{dy} + N(x,y) = 0$ is exact iff $N_x = M_y$

(28) page 101 $\quad y\,dx + (2xy - e^{-2y})\,dy = 0 \quad \| \quad \frac{dy}{dx} = \frac{y}{e^{-2y} - 2xy}$

(**) $\quad y + (2xy - e^{-2y})\frac{dy}{dx} = 0$

$$M = y \qquad N = 2xy - e^{-2y}$$

$$M_y = 1 \qquad N_x = 2y$$

$$\frac{N_x - M_y}{M} = \frac{2y-1}{y} \quad \text{is a function of } y \text{ alone}$$

$$\frac{d\ln\mu}{dy} = \frac{2y-1}{y} = 2 - \frac{1}{y}$$

$$\ln\mu = 2y - \ln y$$

$$\mu = e^{2y - \ln y} = \frac{1}{y}e^{2y}$$

Multiplying (**) by μ :

(***) $\quad e^{2y} + \left(2xe^{2y} - \frac{1}{y}\right)\frac{dy}{dx} = 0$

$$M = e^{2y} \qquad N = 2xe^{2y} - \frac{1}{y}$$

$$M_y = 2e^{2y} \qquad N_x = 2e^{2y}$$

$\therefore$ (***) is exact.

$$\frac{\partial F}{\partial x} = M = e^{2y} \Rightarrow F(x,y) = \int e^{2y}\,dx + g(y)$$

$$= xe^{2y} + g(y)$$

$$\frac{\partial F}{\partial y} = N \Rightarrow x2e^{2y} + g'(y) = 2xe^{2y} - \frac{1}{y}$$

$$g'(y) = -\frac{1}{y} \Rightarrow g(y) = -\ln|y|$$

$$F(x,y) = xe^{2y} - \ln|y| = C$$

or

$$x = \frac{C + \ln|y|}{e^{2y}} = (C + \ln|y|)e^{-2y}$$

Homework for Section 2.6

Page 100
#8
#10
#13
#20 (Note this is not an integrating factor to be found by using Page 19)
#28

Homework for Section 2.7

Page 109
#4
#12

Problems 1-32 from Pages 132-133
~~How to approach these:~~

Linear	Separable	Exact	Homogeneous	Bernoulli
1	2	3	26	32
4	7	5	or 28	
6	10	9	29	
9	13	11		
12	15	14		
17	20	16		
18	22	19		
23		21		
28		24		
27 (subst. $v = x^2$)		25		
		30 (after finding integrating factor)		
		31 "		