MAT 311 Key To Test 2

(4.) $|f(x)-f(y)| = \left|\frac{x}{x^2+1} - \frac{y}{y^2+1}\right| = \left|\frac{yx^2+y-xy^2-x}{(x^2+1)(y^2+1)}\right| = \frac{|xy(x-y)+y-x|}{(x^2+1)(y^2+1)}$

$\leq |xy+1|\,|x-y| \leq 2|x-y|$ if $x,y \in [-1,0]$

$\therefore$ Given $\epsilon > 0$ choose $\delta = \epsilon/2$

$\therefore$ if $x,y \in [-1,0]$ and $|x-y| < \delta$, then $|f(x)-f(y)| \leq 2|x-y|$

$< 2\delta = 2\cdot\frac{\epsilon}{2} = \epsilon.$

(5.) $\forall n \in \mathbb{N}$, let $x_n = -1+\frac{1}{n}$ and $f(x_n) = \frac{-1+\frac{1}{n}}{-1+\frac{1}{n}+1} = -n+1$

Set $\epsilon = 1$, then given $\delta > 0$ choose $\frac{1}{N} < \delta$ $N \in \mathbb{N}$ by Arch. Property

$\therefore$ $\forall n, m,\ n \neq m \geq N$, $|f(x_n)-f(x_m)| = |-n+1-(-m+1)|$

$= |m-n| \geq 1.$

$\therefore$ f is not unif. cont. on $(-1,0)$.

(6) Define $x_k = \frac{\pi}{2} + 2k\pi$, $y_k = \frac{3\pi}{2} + 2k\pi$

$f(x_k) = \frac{\pi}{2} + 2k\pi$ and $f(y_k) = -\left(\frac{3\pi}{2} + 2k\pi\right)$

Set $v \in \mathbb{R}$. $\exists\, k \in \mathbb{N}$ $\ni$ $-\left(\frac{3\pi}{2}+2k\pi\right) < v < \frac{\pi}{2} + 2k\pi$ by Arch. Property

f is cont. on $\left[\frac{\pi}{2}+2k\pi, \frac{3\pi}{2}+2k\pi\right]$

and $f(y_k) < v < f(x_k)$

$\therefore$ By IVT $\exists\, \alpha \in (x_k, y_k)$ $\ni$ $f(\alpha) = v$.

$\therefore$ the range of f is the entire real line.

(4) Given $\epsilon = \frac{1}{2}$ $\exists\, K_1 > 0 \ni \forall x \geq K_1,$ $f(x) \in (L - \frac{1}{2}, L + \frac{1}{2})$

Also

given $\epsilon = \frac{1}{2}$ $\exists\, K_2 < 0 \ni \forall x \leq K_2$ $f(x) \in (2L - \frac{1}{2}, 2L + \frac{1}{2})$

$\therefore$ By EVT, $\exists\ \alpha \in [K_2, K_1]$ such that $\forall x \in [K_2, K_1]$, $f(x) \leq f$

$\therefore$ $\forall x \in \mathbb{R}$ either $x \leq K_2$ or $x \in [K_2, K_1]$ or $x \geq K_1$

$\therefore$ $f(x) \leq 2L + \frac{1}{2}$, $f(x) \leq f(\alpha)$ or $f(x) \leq L + \frac{1}{2}$.

$\therefore$ $\forall x \in \mathbb{R},$ $f(x) \leq \max\{2L + \frac{1}{2}, f(\alpha), L + \frac{1}{2}\} = M$

$\therefore$ All real ~~number~~ numbers greater than M is not in the range of f.

(8)

control	limit: L	$+\infty$	$-\infty$
$x \to c$			
$x \to c^-$			
$x \to c^+$			
$x \to +\infty$			
$x \to -\infty$			

(9) $p(x) = x^4 - x^2 - 1$ is a polynomial and hence continuous on $\mathbb{R}$

$P(-2) = 16 - 4 - 1 = 11$

$P(0) = -1$

$p(2) = 16 - 4 - 1 = 11$

$\therefore$ By IVT $\exists$ zeros between $(-2, 0)$ and $(0, 2)$.

(10) $\lim\limits_{x \to 4} \frac{x-4}{x^2-16} = \lim\limits_{x \to 4} \frac{1}{x+4} = \frac{1}{4+4} = \frac{1}{8}$

Define $f(4)$ to be $\frac{1}{8}$ to ensure $\lim\limits_{x \to 4} f(x) = f(4)$.