MAT 515 Final Spring 2012 Name: Key

Directions: Do at least 10 of the following 12 problems. Each problem is worth 10 points. Extra credit will be available for work beyond 10 problems. Turn in your signed yellow note sheet with this exam.

1. Let f(z) = u +iv and suppose f'(a) exists. Derive the Cauchy Riemann equations at a.

$$f'(a) = \lim_{h\to 0} \frac{u(a_1+h, a_2+0) + iv(a_1+h, a_2+0) - (u(a_1,a_2) + iv(a_1,a_2))}{h}$$

$$= \lim_{h\to 0} \frac{u(a_1+h,a_2) - u(a_1,a_2)}{h} + i\,\frac{v(a_1+h,a_2) - v(a_1,a_2)}{h}$$

$$= u_x(a_1,a_2) + i v_x(a_1,a_2)$$

$$= \lim_{ih\to 0} \frac{u(a_1, a_2+h) + iv(a_1, a_2+h) - (u(a_1,a_2) + iv(a_1,a_2))}{ih}$$

$$= \lim_{ih\to 0} \frac{u(a_1,a_2+h) - u(a_1,a_2) + i(v(a_1,a_2+h) - v(a_1,a_2))}{ih}$$

$$= \frac{u_y(a_1,a_2) + i v_y(a_1,a_2)}{i} = v_y(a_1,a_2) - i u_y(a_1,a_2)$$

$\therefore u_x = v_y$ and $v_x = -u_y$ at (a_1, a_2)

2. Show u = x +e^{x}cos(y) is harmonic. Find the harmonic conjugate of u. What is f'(x+iy)?

$u_x = 1 + e^x \cos y \qquad u_y = -e^x \sin(y)$

$u_{xx} = e^x \cos y \qquad u_{yy} = -e^x \cos(y)$

$u, u_x, u_y, u_{xx}, u_{yy}$ all continuous (including u_{xy} & u_{yx})

and $u_{xx} + u_{yy} = 0 \Rightarrow u$ is harmonic.

$u_x = 1 + e^x \cos y = v_y \Rightarrow v = y + e^x \sin y + g(x)$

$v_x = e^x \sin y + g'(x) = -u_y = e^x \sin y$

$g'(x) = 0$

$g(x) = c$

$\boxed{v = y + e^x \sin y + c}$

3. Find all solutions to $z^5 = 1+i$.

$$z^5 = \sqrt{2}\, e^{i\frac{\pi}{4}} = 2^{\frac{1}{2}} e^{i\left(\frac{\pi}{4} + 2k\pi\right)} \qquad 0 \le k \le 4$$

$$z = 2^{\frac{1}{10}} e^{i\left(\frac{\pi}{20} + \frac{2k\pi}{5}\right)}$$

$$2^{\frac{1}{10}} e^{i\frac{\pi}{20}}$$

$$2^{\frac{1}{10}} e^{i\left(\frac{\pi}{20} + \frac{2}{5}\pi\right)}$$

$$2^{\frac{1}{10}} e^{i\left(\frac{\pi}{20} + \frac{4}{5}\pi\right)}$$

$$2^{\frac{1}{10}} e^{i\left(\frac{\pi}{20} + \frac{6}{5}\pi\right)}$$

$$2^{\frac{1}{10}} e^{i\left(\frac{\pi}{20} + \frac{8}{5}\pi\right)}$$

4. Let $f(z) = z^2e^{-2z}$. Use Taylor's Theorem to determine $f^{(10)}(0)$.

$$f(z) = z^2 \sum_{k=0}^{\infty} \frac{1}{k!}(-1)^k (2z)^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}(-1)^k 2^k z^{k+2}$$

$$k=8 \qquad \frac{1}{8!}(-1)^8 2^8 = \frac{f^{(10)}(0)}{10!}$$

$$\boxed{f^{(10)}(0) = 10 \cdot 9\, (-1)^8 2^8}$$

5. Calculate the following:
   a. The integral of $z\sin(z^2)$ along the straight line from 1+2i to 3.

$F(z) = \frac{1}{2}\sin(z^2)$

$F(3) - F(1+2i) = \frac{1}{2}\sin(9) - \frac{1}{2}\sin((1+2i)^2)$

   b. The integral of $z\sin(z^2)$ along any simple closed contour.

0 by Cauchy-Goursat.

6. Calculate the integral of $e^z/((z^2+4)(z^2+36))$ counterclockwise around the circle of radius 5 centered at 0.

$\dfrac{e^z}{(z-2i)(z+2i)(z^2+36)}$

Residue at $2i$ $\dfrac{e^{2i}}{4i(32)}$

Residue at $-2i$ $\dfrac{e^{-2i}}{-4i(32)}$

Answer: $2\pi i\left[\dfrac{e^{2i}}{128i} - \dfrac{e^{-2i}}{128i}\right] = \dfrac{e^{2i} - e^{-2i}}{\pi 64}$

or $= \dfrac{2i\sin(2)}{\pi 64}$

or $= i\dfrac{\sin(2)}{\pi 32}$

7. State and prove ~~Louiville~~ Liouville's Theorem.

Suppose $f(z)$ is entire and bounded by M, i.e. $|f(z)| \le M$ for all z. Then $f(z)$ must be a constant.

By Cauchy's Inequalities, $|f'(z)| \le \frac{M}{r}$ where

Let $r \to 0$ $\therefore$ $|f'(z)| = 0$

$\therefore$ $f'(z) = 0$

$\therefore$ $f(z)$ must be a constant.

8. State and prove the Fundamental Theorem of Algebra.

Every polynomial equation $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, $n \ge 1$, $a_n \ne 0$ has at least one ~~r~~ root.

If $P(z)$ has no root, then $f(z) = \frac{1}{P(z)}$ is entire and bounded $\left|\frac{1}{P(z)}\right| \le \frac{1}{|a_n||z^n| - \sum_{k=0}^{n-1} |a_k||z^k|} \to 0$ as $|z| \to \infty$

$\therefore$ $\frac{1}{P(z)}$ is a constant by Liouville

$\therefore$ $P(z)$ is a constant which is a contradiction

$\therefore$ $P(z)$ has a root.

9. Show that the absolute convergence of a series of complex numbers implies the convergence of the series.

Assume $\sum_{n=0}^{\infty} |z_n|$ converges, then the remainder term $\sum_{n=j+1}^{\infty} |z_n|$ goes to zero as $j \to +\infty$

$\therefore \left|\sum_{n=0}^{k+1} z_n - \sum_{n=0}^{j} z_n\right| = \left|\sum_{n=j+1}^{k+1} z_n\right| \le \sum_{n=j+1}^{k+1} |z_n| \le \sum_{n=j+1}^{\infty} |z_n|$

$\therefore S_j = \sum_{n=0}^{j} z_n$ is Cauchy

$\therefore S_j$ converges as $j \to +\infty$.

Or using $|x_n|, |y_n| \le |z_n|$, then $\sum |z_n|$ converging implies $\sum |x_n|$ & $\sum |y_n|$ converges. Thus $\sum x_n$ and $\sum y_n$ converges (as real absolutely convergent ~~sequence~~ series)

$\therefore \sum_{n=0}^{\infty} z_n = \sum_{n=0} x_n + i \sum_{n=0} y_n$ converges.

10. Find a Laurent series for $f(z) = (3z-3)/((2z-1)(z-2))$ about $z = 1$ that is convergent for $.5 < |z-1| < 1$.

$\frac{3z-3}{(2z-1)(z-2)} = \frac{A}{2z-1} + \frac{B}{z-2}$; $3z-3 = A(z-2) + B(2z-1)$; let $z = \frac{1}{2}$, $-\frac{3}{2} = A(-\frac{3}{2}) \Rightarrow A = 1$

let $z = 2$, $3 = B(3) \Rightarrow B = 1$

$= \frac{1}{2z-1} + \frac{1}{z-2}$

$\frac{1}{z-2} = \frac{1}{z-1-1} = \frac{-1}{1-(z-1)} = (-1)\sum_{k=0}^{\infty}(z-1)^k$ valid for $|z-1|<1$

$\frac{1}{2z-1} = \frac{1}{2z-2+1} = \frac{1}{2(z-1)} \frac{1}{1+\frac{1}{2(z-1)}} = \frac{1}{2(z-1)} \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{2(z-1)}\right)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} (z-1)^{-k-1}$

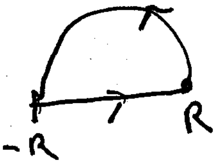
valid for $\frac{1}{2|z-1|} < 1$ or $.5 < |z-1|$

$j = -(k+1)$; $-j = (k+1)$; $-j-1 = k$

$\therefore \boxed{\frac{3z-3}{(2z-1)(z-2)} = \sum_{j=-1}^{-\infty} (-1)^{-j-1} 2^{j} (z-1)^{j} + \sum_{k=0}^{\infty} (-1)(z-1)^k}$

valid for $.5 < |z-1| < 1$

11. Calculate P.V. $\int_{-\infty}^{+\infty} \frac{1}{(x^2+2x+2)(x^2+1)^2}dx$.



C_r : $|z| = R$ in upper half plane

$\gamma_r = [-R, R] \cup C_R$ counterclockwise

$$\left|\int_{C_R} \frac{1}{(z^2+2z+2)(z^2+1)^2}\,dz\right| \le \frac{\pi R}{(R^2-2R-2)(R^2-1)^2} \to 0 \text{ as } R \to +\infty.$$

$$\frac{1}{z^2+2z+2}\,\frac{1}{(z^2+1)^2} = \frac{1}{((z+1)^2+1)(z^2+1)^2} = \frac{1}{(z+1-i)(z+1+i)(z-i)^2(z+i)^2}$$

For large R, $\int_{\gamma_R} \frac{1}{(z^2+2z+2)(z^2+1)^2}\,dz = 2\pi i\left[\operatorname{Res}_{z=i} f + \operatorname{Res}_{z=-1+i} f\right]$

simple pole at $z=-1+i$: $\operatorname{Res}_{z=-1+i} f = \frac{1}{(-1+i+1+i)((-1+i)^2+1)^2} = \frac{1}{2i(1-2i-1+1)^2} = \frac{1}{2i(1-2i)^2}$

$$= \frac{1}{2i(1-4i-4)} = \boxed{\frac{-3+4i}{2i(25)}}$$

pole of order 2 at $z=i$: $\operatorname{Res}_{z=i} f = \lim_{z\to i} \frac{d}{dz}\left((z^2+2z+2)(z+i)^2\right)^{-1}$

$= \lim_{z\to i} (-1)\left((z^2+2z+2)(z+i)^2\right)^{-2}\left\{(2z+2)(z+i)^2+(z^2+2z+2)2(z+i)\right\}$

$= (-1)\left[(1+2i)(2i)^2\right]^{-2}\left[2(i+1)(2i)^2+(1+2i)2(2i)\right]$

$= (-1)\left[(1+2i)(-4)\right]^{-2}4\left[2(i+1)(-1)+(-2+i)\right]$

$= -\frac{1}{4}\,\frac{2i+2+2-i}{(1+2i)^2} = \frac{1}{4}\,\frac{4+i}{(-3+4i)} = \frac{1}{4}\,\frac{(4+i)(-3-4i)}{(25)} = \boxed{\frac{1}{4(25)}(-8-19i)}$

$\therefore 2\pi i\left[\operatorname{Res}_{z=-1+i} f + \operatorname{Res}_{z=i} f\right] = 2\pi i\left[\frac{-3+4i}{2i(25)} + \frac{1}{4(25)}(-8-19i)\right] = \frac{\pi}{50}(-6+8i) + \frac{\pi}{50}(-8i+19)$

$$= \boxed{\frac{\pi}{50}(13)}$$

12. Suppose f(z) = $\sum_{n=0}^{\infty} \frac{2z^{2n+1}}{2n+1}$ is valid for |z|< 1. Calculate the integral of 1/f(z) counterclockwise around the circle of radius ½ centered at 0.

Assume $f(z)=0$ only at $z=0$ inside the circle of radius $\frac{1}{2}$ centered at 0.

Then $\frac{1}{f(z)} = \frac{1}{z\sum_{n=0}^{\infty}\frac{2z^{2n}}{2n+1}} = \frac{1}{z}\phi(z)$ where $\phi(z) = \frac{1}{2+\sum_{n=1}^{\infty}\frac{2z^{2n}}{2n+1}}$

$$\therefore \int_{|z|=\frac{1}{2}} \frac{1}{f(z)}\,dz = \int_{|z|=\frac{1}{2}} \frac{\phi(z)}{z-0} = 2\pi i\,\phi(0) = 2\pi i\left(\frac{1}{2}\right)$$

$$= \boxed{\pi i}$$