MAT 361 Key To Test 2

① $y'' + 2y' - 15y = 0$

$r^2 + 2r - 15 = 0$

$(r+5)(r-3)$

$y(t) = c_1 e^{-5t} + c_2 e^{3t}$

② $y'' + 4y' + 5y = e^t$

$r^2 + 4r + 5 = 0$

$(r+2)^2 + 1 = 0$

$(r+2)^2 = -1$

$r + 2 = \pm i$

$r = -2 \pm i$

$y_h(t) = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t$

$y_p(t) = Ae^t \quad y_p'(t) = Ae^t \quad y_p'' = Ae^t$

$e^t = y'' + 4y' + 5y = 10Ae^t \Rightarrow A = \frac{1}{10}$

$y_p(t) = \frac{1}{10}e^t$.

$\therefore \boxed{y(t) = \frac{1}{10}e^t + c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t}$

③ $y'' + 6y' + 9y = t$

$y'' + 6y' + 9y = 0$

$(r+3)^2 = 0$

$y_h(t) = c_1 e^{-3t} + c_2 t e^{-3t}$

$y_p(t) = A + Bt \qquad y_p'' + 6y_p' + 9y_p = 6B + 9(A+Bt) = t$

$y_p'(t) = B \qquad 6B + 9A = 0 \quad 9B = 1$

$y_p''(t) = 0 \qquad B = \frac{1}{9} \quad 9A = -6B = -\frac{6}{9} = -\frac{2}{3}$

$A = -\frac{2}{27} \quad y_p(t) = -\frac{2}{27} + \frac{1}{9}t$

$\boxed{y(t) = -\frac{2}{27} + \frac{1}{9}t + c_1 e^{-3t} + c_2 t e^{-3t}}$

4. Use the method of annihilators to solve $y''-y'+.25y = 4e^{.5t}$

$r^2 - r + \frac{1}{4} = 0$

$(r-\frac{1}{2})^2 = 0$

$\therefore$ $(D-\frac{1}{2})^2$ annihilates $y''-y'+.25y$

and $(D-\frac{1}{2})$ annihilates $4e^{.5t}$

$(D-\frac{1}{2})^3$ annihilates ~~general solution~~ $y''-y'+.25y$ and $4e^{.5t}$

particular solution is $y_p(t) = At^2e^{.5t}$

$y_p'(t) = e^{.5t}(\frac{1}{2}At^2 + 2At)$

$y_p''(t) = e^{.5t}(\frac{1}{4}At^2 + At + At + 2A)$

$= e^{.5t}(\frac{1}{4}At^2 + 2At + 2A)$

$y_p''(t) - y'(t) + .25y = e^{.5t}(\cancel{\frac{1}{4}At^2} + \cancel{2At} + 2A - \cancel{\frac{1}{2}At^2} - \cancel{2At} + \cancel{\frac{1}{4}At^2})$

$= 4e^{.5t}$

$2A = 4 \Rightarrow A = 2$

$y_p(t) = 2t^2e^{.5t}$

$$\boxed{y(t) = 2t^2e^{.5t} + c_1e^{.5t} + c_2te^{.5t}}$$

(5) Use variation of parameters to solve $y^{(3)} - 4y' = 1$

$$y^{(3)} - 4y' = 0$$
$$r^3 - 4r = 0$$
$$r(r-2)(r+2)$$
$$y_1(t) = 1 \quad y_2(t) = e^{-2t} \quad y_3(t) = e^{2t}$$

$$W(y_1, y_2, y_3) = \begin{vmatrix} 1 & e^{-2t} & e^{2t} \\ 0 & -2e^{-2t} & 2e^{2t} \\ 0 & 4e^{-2t} & 4e^{2t} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 4 & 4 \end{vmatrix}$$

$$= -8 - 8 = -16$$

$$u_1' = \frac{\begin{vmatrix} 0 & e^{-2t} & e^{2t} \\ 0 & -2e^{-2t} & 2e^{2t} \\ 1 & 4e^{-2t} & 4e^{2t} \end{vmatrix}}{-16} = \frac{\begin{vmatrix} 0 & 1 & 1 \\ 0 & -2 & 2 \\ 1 & 4 & 4 \end{vmatrix}}{-16} = \frac{2+2}{-16} = \frac{4}{-16} = -\frac{1}{4}$$

$$u_1 = -\frac{1}{4}t$$

$$u_2'(t) = \frac{\begin{vmatrix} 1 & 0 & e^{2t} \\ 0 & 0 & 2e^{2t} \\ 0 & 1 & 4e^{2t} \end{vmatrix}}{-16} = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 4 \end{vmatrix} e^{2t}}{-16} = \frac{(-2)e^{2t}}{-16} = \frac{1}{8}e^{2t}$$

$$u_2(t) = \frac{1}{16}e^{2t}$$

$$u_3'(t) = \frac{\begin{vmatrix} 1 & e^{-2t} & 0 \\ 0 & -2e^{-2t} & 0 \\ 0 & 4e^{-2t} & 1 \end{vmatrix}}{-16} = \frac{e^{-2t}\begin{vmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 4 & 1 \end{vmatrix}}{-16} = \frac{e^{-2t}(-2)}{-16} = \frac{1}{8}e^{-2t}$$

$$u_3(t) = -\frac{1}{16}e^{-2t}$$

$$y_p(t) = u_1y_1 + u_2y_2 + u_3y_3 = -\frac{1}{4}t + \frac{1}{16} - \frac{1}{16} = -\frac{1}{4}t$$

general solution is

$$\boxed{y(t) = -\frac{1}{4}t + c_1 + c_2e^{-2t} + c_3e^{2t}}$$

6

$y'' - 2ty' - 2y = 0$

$y(t) = \sum_{n=0}^{\infty} a_n t^n$ $\qquad$ $-2y(t) = \sum_{n=0}^{\infty} (-2) a_n t^n$

$y'(t) = \sum_{n=1}^{\infty} a_n n t^{n-1}$ $\qquad$ $-2ty'(t) = \sum_{n=1}^{\infty} (-2) a_n n t^n$

$y''(t) = \sum_{n=2}^{\infty} a_n n(n-1) t^{n-2}$ $\qquad$ $y''(t) = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1) t^n$

$y''(t) - 2ty'(t) - 2y(t) = 0 = -2a_0 + a_2(2)(1) + \sum_{n=1}^{\infty} \left[a_{n+2}(n+2)(n+1) - 2a_n n - 2a_n\right] t^n$

$-2a_0 + 2(a_2) \Rightarrow a_2 = a_0$

For $n \geq 1$, $a_{n+2}(n+2)(n+1) + 2a_n(n+1) = 0 \Rightarrow a_{n+2} = \frac{2\, a_n}{n+2}$

$y_1(t)$ soln of $y_1(0) = 1$, $y_1'(0) = 0$

$a_0 = 1$ $\qquad$ $a_1 = 0 \Rightarrow a_{2k+1} = 0.$

$a_2 = \frac{2}{2} a_0 = 1$

$a_4 = \frac{2}{4}(1) = \frac{1}{2}$

$a_6 = \frac{2}{6}\left(\frac{1}{2}\right) = \frac{1}{6} = \frac{1}{3 \cdot 2}$

$a_8 = \frac{2}{8} \frac{1}{3 \cdot 2} = \frac{1}{2 \cdot 3 \cdot 4}$

$a_{10} = \frac{2}{10} \frac{1}{2 \cdot 3 \cdot 4} = \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}$

$a_{2k} = \frac{1}{k!}$

$\therefore y_1(t) = e^{t^2}$

radius of convergence $= 1$

$y_2(t)$ soln of $y_2(0) = 0$, $y_2'(0) = 1$

$a_0 = 0 \Rightarrow a_{2k} = 0$ $\qquad$ $a_1 = 1$

$a_3 = \frac{2}{3} = \frac{2^2}{3!} = \frac{2^2 1!}{3!}$

$a_5 = \frac{2}{5} \frac{2^2}{3!} = \frac{2^3 \cdot 4}{5!} = \frac{2^4 \cdot 2!}{5!}$

$a_7 = \frac{2}{7} \frac{2^3 \cdot 4 \cdot 6}{6 \cdot 5!} = \frac{2^6 \, 3!}{7!}$

$a_9 = \frac{2}{9} \frac{2^6 \, 3!}{7!} \frac{4 \cdot 2}{8} = \frac{2^8 \, 4!}{9!}$

$a_{11} = \frac{2}{11} \frac{2^8 \cdot 4!}{9!} = \frac{2 \, 2^8 \cdot 4! \cdot 2 \cdot 5}{10!} = \frac{2^{10} \, 5!}{11!}$

$a_{13} = \frac{2}{13} \frac{2^{10} \cdot 5!}{11!} = \frac{2 \cdot 2^{10} \cdot 5! \cdot 6 \cdot 2}{13!} = \frac{2^{12} \, 6!}{13!}$

$a_{2k+1} = \frac{2^{2k} \, k!}{(2k+1)!}$; $y_2(t) = \frac{\sqrt{\pi}}{2} \operatorname{erf}(t) e^{t^2}$

where $\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} ds$