Test 3 Key Attached

I. For example 5, page 309

show why $\int_0^{+\infty} e^{-(s-a)t}\,dt = \frac{1}{s-a}$

by computing directly the improper integral.

In other words, do not use #2 on page 317.

II. Use #5, #6, + #14 on page 317 to prove #9 + #10 on page 317

(In other words, do not use the hints about problems 13 + 14 on page 311.)

III. Find the inverse Laplace Transform of

$$R(s) = \frac{s^2+1}{s^3-2s^2-8s}$$

Hint: See last half of Ex 1 on page 315 and use #1 + #2 on page 317

IV. Solve using Laplace transforms

$$y'' - 2y' - 8y = 1, \quad y(0) = 1, \quad y'(0) = 2$$

(See bottom of page 315).

I. $\int_0^{+\infty} e^{-(s-a)t}\,dt = \lim_{b\to+\infty} \int_0^{b} e^{-(s-a)t}\,dt$ $\qquad u = (s-a)t$

$\frac{du}{dt} = s-a$

$= \lim_{b\to+\infty} \int_0^{(s-a)b} e^{-u}\frac{1}{s-a}\,du$ $\qquad du = (s-a)\,dt$

$= \lim_{b\to+\infty} \frac{1}{s-a}\left(\frac{e^{-u}}{-1}\right)\Big|_0^{(s-a)b}$

$= \lim_{b\to+\infty} \frac{1}{s-a}\left(\frac{e^{-(s-a)b}}{-1} + \frac{e^{-\cancel{(s-a)b}\,0}}{1}\right)$

$= \lim_{b\to+\infty} \frac{1}{s-a}\left(1 - e^{-(s-a)b}\right)$

$= \frac{1}{s-a}(1-0) = \frac{1}{s-a}.$

II. $\mathcal{L}\{e^{at}\sin bt\} = F(s-a)$ when $F(s) = \mathcal{L}\{\sin bt\}$

$= \frac{b}{s^2+b^2}$

$= \frac{b}{(s-a)^2+b^2}$

$\mathcal{L}\{e^{at}\cos bt\} = F(s-a)$ where $F(s) = \mathcal{L}\{\cos bt\}$

$= \frac{s}{s^2+b^2}$

$= \frac{s-a}{(s-a)^2+b^2}$

III. $R(s) = \frac{s^2+1}{s^3-2s^2-8s} = \frac{s^2+1}{s(s^2-2s-8)} = \frac{s^2+1}{s(s-4)(s+2)}$

$= \frac{A}{s} + \frac{B}{s-4} + \frac{C}{s+2}$

$s^2+1 = A(s-4)(s+2) + Bs(s+2) + Cs(s-4)$

s^2 | $1 = A + B + C$

s | $0 = -2A + 2B - 4C$

1 | $1 = -8A$

$A = -\frac{1}{8}$

$4 = 4A + 4B + 4C$

$0 = -2A + 2B - 4C$

$4 = 2A + 6B$

$4 = -\frac{1}{4} + 6B$

$\frac{4+\frac{1}{4}}{6} = 0 = 6B \Rightarrow B = \frac{17}{36}$

$s^2 \quad 1 = A + B + C$

$s \quad 0 = -2A + 2B - 4C$

$1 \quad 1 = -8A \Rightarrow A = -\frac{1}{8}$

$\left.\begin{aligned} \frac{9}{8} &= B + C \\ -\frac{1}{4} &= 2B - 4C \end{aligned}\right\} \quad \begin{aligned} -\frac{9}{4} &= -2B - 2C \\ -\frac{1}{4} &= 2B - 4C \end{aligned}$

$-\frac{5}{2} = -6C$

$C = \frac{5}{12}$

$B = 1 - A - C = 1 - (-\frac{1}{8}) - \frac{5}{12} = \frac{9}{8} - \frac{5}{12} = \frac{27}{24} - \frac{10}{24} = \frac{17}{24}$

$R(s) = -\frac{1}{8}\frac{1}{s} + \frac{17}{24}\frac{1}{s-4} + \frac{5}{12}\frac{1}{s+2}$

$y(t) = -\frac{1}{8} + \frac{17}{24}e^{4t} + \frac{5}{12}e^{-2t}$

IV. (14) $ay'' + by' + cy = f(t)$

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}$$

$y'' - 2y' - 8y = 1 \qquad y(0) = 1 \qquad y'(0) = 2$

$a = 1, b = -2, c = -8 \qquad F(s) = \frac{1}{s}$

$$Y(s) = \frac{(s-2)(1) + 2}{s^2 - 2s + 8} + \frac{\frac{1}{s}}{s^2 - 2s + 8}$$

$$= \frac{s[(s-2) + 2]}{s(s^2 - 2s + 8)} + \frac{1}{s^2 - 2s + 8}$$

$$= \frac{s^2 + 1}{s^3 - 2s^2 + 8s}$$

from #3

$y(t) = -\frac{1}{8} + \frac{17}{24}e^{4t} + \frac{5}{12}e^{-2t}$