Section 2.1

$$\frac{dy}{dt} + p(t)\,y = g(t)$$

$$\mu(t)\frac{dy}{dt} + \mu(t)\,p(t)\,y = \mu(t)\,g(t)$$

to force

$$\frac{d}{dt}\big(\mu(t)\,y(t)\big) = \mu(t)\frac{dy}{dt} + \mu(t)\,p(t)\,y$$

$$\mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}\,y(t) = \quad "$$

$$\therefore \quad \frac{d\mu}{dt} = \mu(t)\,p(t)$$

$$\frac{1}{\mu(t)}\frac{d\mu}{dt} = p(t)$$

$$\ln \mu(t) = \int p(t)\,dt$$

$$\mu(t) = e^{\int p(t)\,dt}$$

Ex: $y(x) = e^x x^2 + x$

$$\frac{dy}{dx} = e^x x^2 + e^x(2x) + 1 = e^x(2x) + 1 + y(x) - x$$

$$\boxed{\frac{dy}{dx} - y(x) = 2e^x x - x + 1}$$

$$\mu(x) = e^{-x}$$

$$\frac{d}{dx}\left(e^{-x}y(x)\right) = 2x - xe^{-x} + e^{-x}$$

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx$$

$u = x \quad dv = e^{-x}\,dx \qquad = -xe^{-x} - e^{-x} + C$

$du = dx \quad v = -e^{-x}$

$$e^{-x}y(x) = x^2 + xe^{-x} \cancel{+e^{-x}} \cancel{-e^{-x}} + C$$

$$\boxed{y(x) = e^x(x^2) + x + Ce^x}$$

$$\frac{dy}{dx} - y(x) = e^x(x^2) + e^x(2x) + 1 + Ce^x \cancel{- Ce^x - e^x(x^2) + x + Ce^x}$$

$$= 2e^x x - x + 1$$

first order linear $\frac{dy}{dt} + p(t)\, y(t) = g(t)$ (3) page 32

Multiply by

integrating factor $\mu(t) = e^{\int p(t)\,dt}$

to obtain $\frac{d}{dt}\left(\mu(t)\, y(t)\right) = \mu(t)\, g(t)$

$$\mu(t)\, y(t) = \int^{t} \mu(s)\, g(s)\, ds$$

(33) page 34 $$y(t) = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)\, g(s)\, ds + c\right]$$

#18 page 40 $t y' + 2y = \sin t, \quad y(\frac{\pi}{2}) = 1, \quad t > 0$

$$y' + \frac{2}{t} y = \frac{1}{t}\sin t$$

$$\mu(t) = e^{\int \frac{2}{t} dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

$$\frac{d}{dt}\left(t^2 y(t)\right) = t \sin t$$

$$t^2 y(t) = \int t \sin t = -t\cos t + \int \cos t\, dt$$

$u = t \quad dv = \sin t$ $\qquad = -t\cos t + \sin t + c$

$du = dt \quad v = -\cos t$

$$\left(\frac{\pi}{2}\right)^2 (1) = -\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} + c$$

$$\frac{\pi^2}{4} = 1 + c \Rightarrow c = \frac{\pi^2}{4} - 1$$

$$t^2 y(t) = -t\cos t + \sin t + \frac{\pi^2}{4} - 1$$

$$y(t) = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{\frac{\pi^2}{4} - 1}{t^2}$$

2.2

Separable differential equations

$$\frac{dy}{dx} = f(x)g(y)$$

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

$$G(y) = F(x) + c$$

#14 $\frac{dy}{dx} = xy^3(1+x^2)^{-\frac{1}{2}}$ $\quad y(0)=1$

(a)

$$\frac{dy}{dx} = \frac{x}{(1+x^2)^{\frac{1}{2}}}\,y^3$$

$$\int \frac{1}{y^3}\,dy = \int \frac{x}{(1+x^2)^{\frac{1}{2}}}\,dx$$

$$-\frac{1}{2}y^{-2} = (1+x^2)^{\frac{1}{2}} + c$$

$$-\frac{1}{2} = 1 + c \Rightarrow c = -\frac{3}{2}$$

$$-\frac{1}{2}\frac{1}{y^2} = (1+x^2)^{\frac{1}{2}} - \frac{3}{2}$$

$$\frac{1}{y^2} = -2(1+x^2)^{\frac{1}{2}} + 3$$

$$y^2 = \frac{1}{3-2(1+x^2)^{\frac{1}{2}}}$$

$$y = \frac{1}{\sqrt{3-2(1+x^2)^{\frac{1}{2}}}}$$

(c)

$$3 - 2(1+x^2)^{\frac{1}{2}} = 0$$

$$\frac{3}{2} = (1+x^2)^{\frac{1}{2}}$$

$$\frac{9}{4} = 1+x^2$$

$$\frac{5}{4} = x^2$$

$$x = \pm\frac{\sqrt{5}}{2} \qquad \left[-\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}\right]$$

Homogeneous Equation

$$\frac{dy}{dx} = f(x,y) = g(v) \quad \text{where } v = \frac{y}{x}$$

$$y = vx$$

$$\frac{dy}{dx} = \frac{dv}{dx}x + v$$

$$\frac{dv}{dx}x + v = g(v)$$

$$\frac{dv}{dx} = \frac{g(v) - v}{x} \quad \text{separable}$$

36 $(x^2 + 3xy + y^2)\,dx - x^2\,dy = 0$

$$\frac{dy}{dx} = 1 + 3\frac{y}{x} + \left(\frac{y}{x}\right)^2 \qquad v = \frac{y}{x}$$

$$= 1 + 3v + v^2$$

$$\therefore \frac{dv}{dx} = \frac{1 + 3v + v^2 - v}{x} = \frac{1 + 2v + v^2}{x}$$

$$\frac{1}{(1+v)^2}\,dv = \frac{1}{x}\,dx$$

$$-\frac{1}{1+v} = \ln|x| + c$$

$$\frac{1}{1+v} = -\ln|x| + C = C - \ln|x|$$

$$1 + v = \frac{1}{C - \ln|x|}$$

$$v = \frac{1}{C - \ln|x|} - 1$$

$$y = vx = x\left(\frac{1}{C - \ln|x|} - 1\right)$$

Section 2.3 $P(t)$ = amount in account at time t

8 $\frac{dP}{dt} = rP + k$, $P(0) = 0$

$\frac{dP}{dt} - rP = k$

$\mu(t) = e^{-rt}$

$\frac{d}{dt}\left(e^{-rt}P(t)\right) = ke^{-rt}$

$e^{-rt}P(t) = -\frac{k}{r}e^{-rt} + C$

at $t=0$ $\quad 0 = -\frac{k}{r} + C \Rightarrow C = \frac{k}{r}$

$P(t) = e^{rt}\left(-\frac{k}{r}e^{-rt} + \frac{k}{r}\right)$

$= \frac{k}{r}\left(-1 + e^{rt}\right)$

Check $P'(t) - rP = \frac{k}{r}\left(re^{rt}\right) - k\left(-1 + e^{rt}\right) = k$

and $P(0) = 0$

(a) $S(t) = \frac{k}{r}\left(e^{rt} - 1\right)$

(b) $S(40) = 1000000$

$\frac{k}{.075}\left(e^{+.075(40)} - 1\right) = 1{,}000{,}000$

$k = \frac{75000}{e^{3} - 1} = \$3929.68$

(c) $S(40) = 1{,}000{,}000$

$\frac{20000}{r}\left(e^{40r} - 1\right) = 1{,}000{,}000$

$\frac{e^{40r} - 1}{r} = 500$ or $e^{40r} = 1 + 500r$

Solve by calculator: $r = 9.773\%$

14 (a) $\frac{dy}{dt} = \frac{(.5 + \sin t)}{5} y \quad , y(0) = y_0$

$\int \frac{1}{y} dy = \int \frac{1}{5}(.5 + \sin t)\, dt$

$\ln|y| = \frac{1}{5}(.5t - \cos t) + C$

$\ln|y_0| = \frac{1}{5}(-1) + C \Rightarrow C = \ln|y_0| + .2$

$\ln|y| = \frac{1}{5}(.5t - \cos t) + \ln|y_0| + .2$

$|y(t)| = |y_0| e^{.2} e^{\frac{1}{5}(.5t - \cos t)}$

Assume $y_0 > 0$ $\quad y(t) = y_0 e^{.2} e^{\frac{1}{5}(.5t - \cos t)} = y_0 e^{.2 - \frac{1}{5}\cos t} e^{.1t}$

Doubling time is $T \ni y(T) = 2y_0$

$2y_0 = y_0 e^{.2} e^{\frac{1}{5}(.5T - \cos T)}$

$2 = e^{.2} e^{\frac{1}{5}(.5T - \cos T)}$

$2e^{-.2} = e^{\frac{1}{5}(.5T - \cos T)} = e^{.1T - \frac{1}{5}\cos t}$

Doubling time does not depend upon the value of y_0

Doubling time is 2.9632084

(b) $\int \frac{1}{y} dy = \int \frac{1}{5}(.1 + \sin t)\, dt$

$\ln|y| = \frac{1}{5}(.1t - \cos t) + C$

$\ln|y_0| = \frac{1}{5}(-1) + C \Rightarrow C = \ln|y_0| + .2$

Assume $y_0 > 0$ $\quad y(t) = y_0 e^{.2} e^{\frac{1}{5}(.1t - \cos t)}$

$= y_0 e^{.2 - \frac{1}{5}\cos t} e^{.02t}$

Doubling time T: $\quad 2 = e^{.2 - \frac{1}{5}\cos T} e^{.02T}$

$2e^{-.2} = e^{.02T - \frac{1}{5}\cos T}$

```
> ode := diff(y(x),x) = (1/245)*(49^2-y(x)^2);
```

$$ode := \frac{d}{dx}\,\mathrm{y}(x) = \frac{49}{5} - \frac{1}{245}\,\mathrm{y}(x)^2$$

```
> dsolve(ode);
```

$$\mathrm{y}(x) = \frac{49\left(\mathbf{e}^{\left(\frac{2\,x}{5}\right)}\,\_C1 + 1\right)}{-1 + \mathbf{e}^{\left(\frac{2\,x}{5}\right)}\,\_C1}$$

```
> ics := y(0) = 0;
```

$$ics := \mathrm{y}(0) = 0$$

```
> dsolve({ode,ics});
```

$$\mathrm{y}(x) = -\frac{49\left(-2\,\mathbf{e}^{\left(\frac{2\,x}{5} + 2\,I\,\pi\,\_Z1\sim - \pi\,I\right)} - 1 + \mathbf{e}^{\left(\frac{2\,x}{5} - \pi\,I\right)}\right)}{-1 + \mathbf{e}^{\left(\frac{2\,x}{5} - \pi\,I\right)}}$$

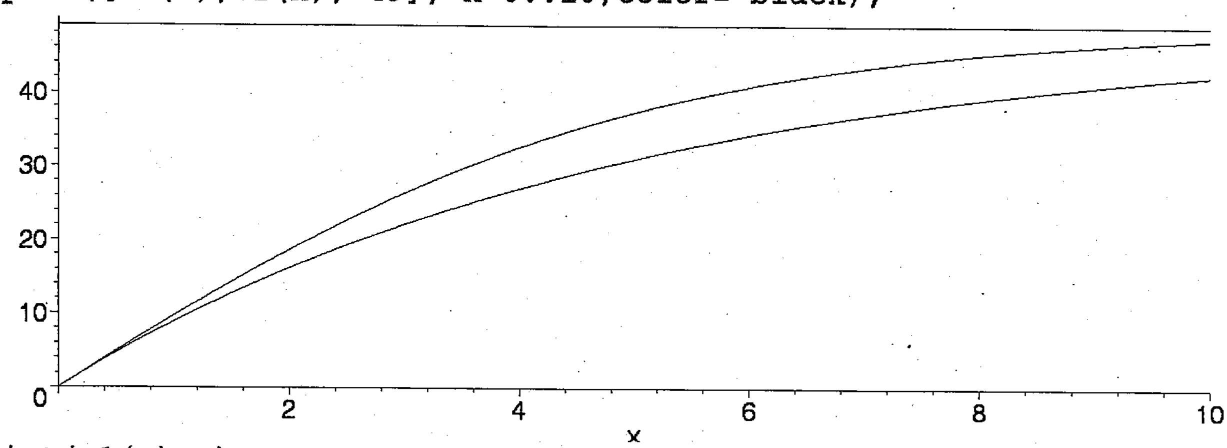
```
> v1:=x->49*(1-exp(-.4*x))/(1+exp(-.4*x));
```

$$v1 := x \rightarrow \frac{49\,(1 - \mathbf{e}^{(-0.4\,x)})}{1 + \mathbf{e}^{(-0.4\,x)}}$$

```
> v2:=x->49*(1-exp(-.2*x));
```

$$v2 := x \rightarrow 49 - 49\,\mathbf{e}^{(-0.2\,x)}$$

```
> plot([v1(x),v2(x), 49], x=0..10,color= black);
```

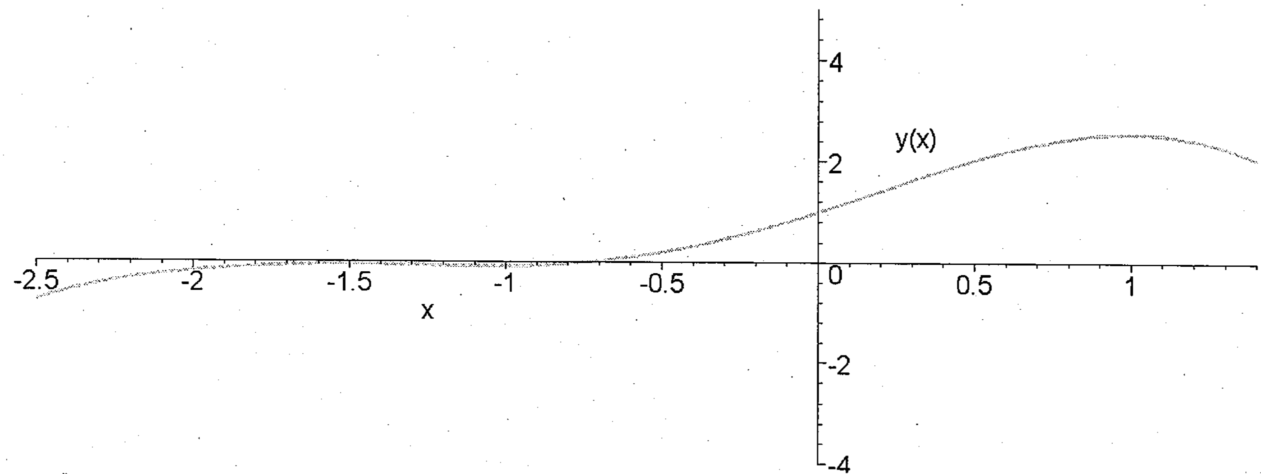



```
> int(v1(x),x);
```

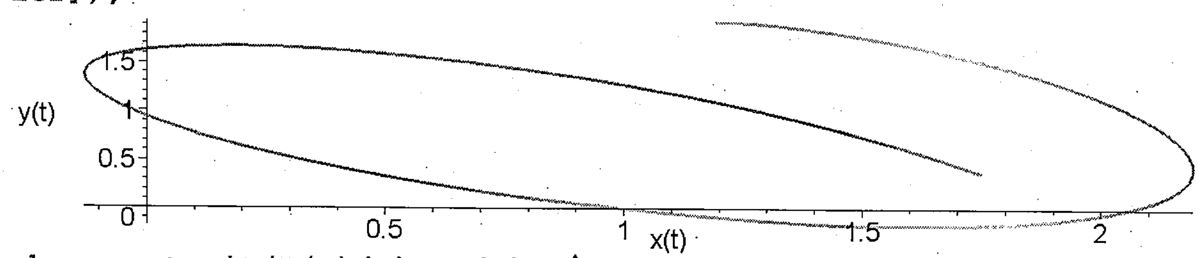
$$245.\,\ln(1. + \mathbf{e}^{(-0.4000000000\,x)}) - 122.5000000\,\ln(\mathbf{e}^{(-0.4000000000\,x)})$$

```
>
```

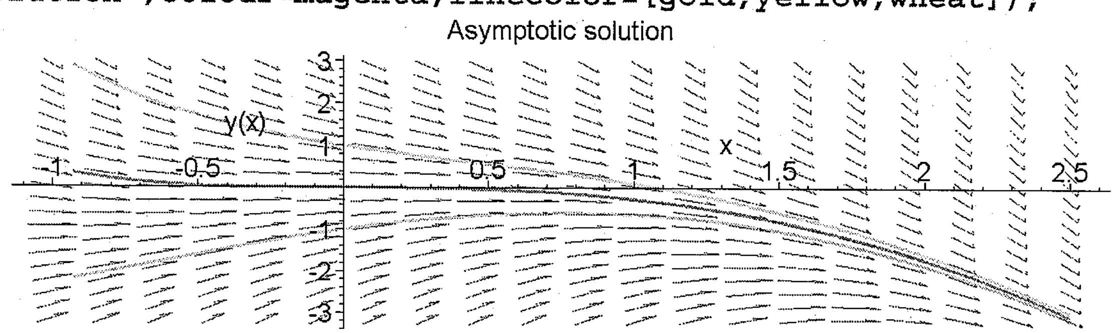
```
[ > with(DEtools):
> phaseportrait(cos(x)*diff(y(x),x$3)-diff(y(x),x$2)+Pi*diff(y(x)
  ,x)=y(x)-x,y(x),x=-2.5..1.4,[[y(0)=1,D(y)(0)=2,(D@@2)(y)(0)=1]]
  ,y=-4..5,stepsize=.05);
```




```
> phaseportrait([D(x)(t)=y(t)-z(t),D(y)(t)=z(t)-x(t),D(z)(t)=x(t)
  -y(t)*2],
  [x(t),y(t),z(t)],t=-2..2,[[x(0)=1,y(0)=0,z(0)=2]],stepsize=.05,
  scene=[x(t),y(t)],linecolour=sin(t*Pi/2),method=classical[foreu
  ler]);
```




```
> phaseportrait(D(y)(x)=-y(x)-x^2,y(x),x=-1..2.5,[[y(0)=0],[y(0)=
  1],[y(0)=-1]],title=`Asymptotic
  solution`,colour=magenta,linecolor=[gold,yellow,wheat]);
```

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[ >
```