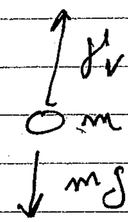
$F = ma$ Newton's Second Law

$F = m\frac{dv}{dt}$

$F = mg - \gamma v$



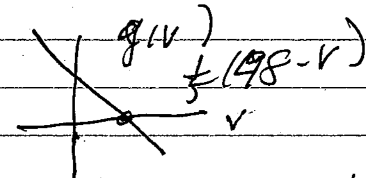
$m\frac{dv}{dt} = mg - \gamma v$

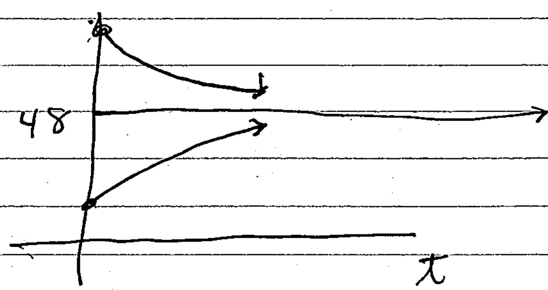
or $\frac{dv}{dt} = g - \frac{\gamma}{m} v$

Suppose $m = 10$ kg $\gamma = 2$ kg/s $g = 9.8$ m/sec^2

$\frac{dv}{dt} = 9.8 - \frac{v}{5} = \frac{1}{5}\left((9.8)5 - v\right)$

(*) $= \frac{1}{5}(48 - v)$, $v(0) = v_0$

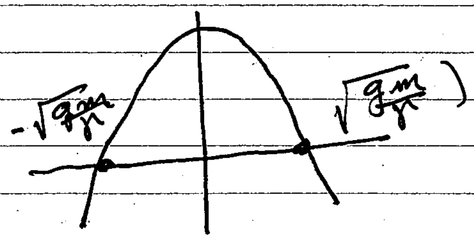


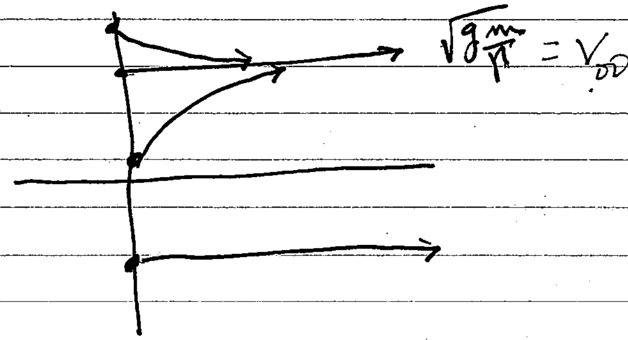


equilibrium solution

(25) $\frac{dv}{dt} = g - \frac{\gamma}{m}v^2 = -\frac{\gamma}{m}\left(v^2 - g\left(\frac{m}{\gamma}\right)\right) = -\frac{\gamma}{m}\left(v - \sqrt{\frac{gm}{\gamma}}\right)\left(v + \sqrt{\frac{gm}{\gamma}}\right)$

$g(v) = -\frac{\gamma}{m}\left(v - \sqrt{\frac{gm}{\gamma}}\right)\left(v + \sqrt{\frac{gm}{\gamma}}\right)$





$\sqrt{\frac{gm}{\gamma}} = 49 = v_\infty$

$\frac{gm}{\gamma} = 49^2$

$\frac{gm}{49^2} = \gamma = \frac{(9.8)(10)}{49^2} = \frac{98}{49^2} = \frac{2}{49}$ kg/sec.

For discussion Friday

Solution to #25, page 9

(25) From (4) on page 2: $m\frac{dv}{dt} = mg - \gamma v$,

the exponent of v changes from 1 to 2.

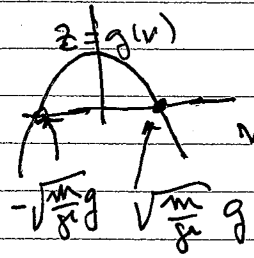
(a) $\therefore\ m\frac{dv}{dt} = mg - \gamma v^2$ or $\boxed{\frac{dv}{dt} = g - \frac{\gamma}{m}v^2}$

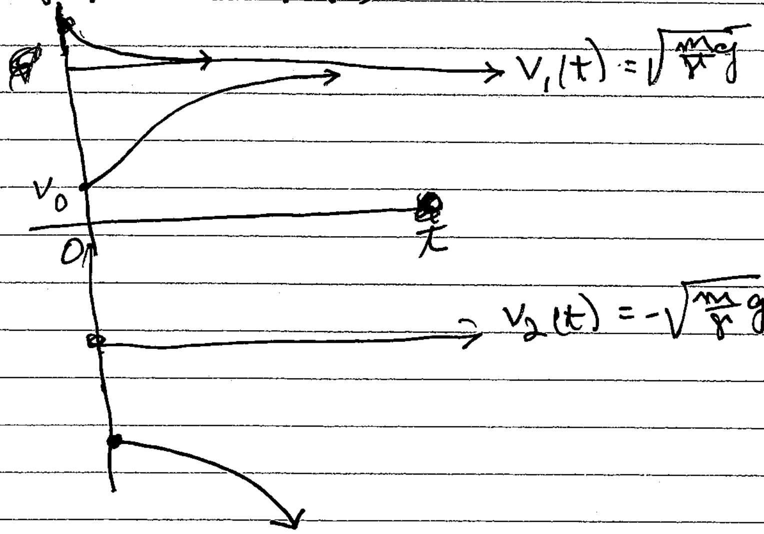
(b) $\frac{dv}{dt} = -\frac{\gamma}{m}\left(v^2 - \frac{m}{\gamma}g\right) = -\frac{\gamma}{m}\left(v - \sqrt{\frac{m}{\gamma}g}\right)\left(v + \sqrt{\frac{m}{\gamma}g}\right)$

$\therefore$ the equilibrium solutions are $v_1(t) = \sqrt{\frac{m}{\gamma}g}$

and $v_2(t) = -\sqrt{\frac{m}{\gamma}g}$

Set $z = g(v) = -\frac{\gamma}{m}\left(v - \sqrt{\frac{m}{\gamma}g}\right)\left(v + \sqrt{\frac{m}{\gamma}g}\right)$





For v_0 where $v_0 > -\sqrt{\frac{m}{\gamma}g}$, $v(t) \to \sqrt{\frac{m}{\gamma}g}$ as $t \to +\infty$.

$\sqrt{\frac{m}{\gamma}g}$ is the limiting velocity.

(c) Let $m = 10$ and solve for γ: $\sqrt{\frac{10}{\gamma}9.8} = 49$

$\frac{10}{\gamma}(9.8) = 49^2$

$\frac{98}{49^2} = \gamma \Rightarrow$

$\gamma = \frac{2}{49}$

Solution to #11, page 17

11 From #25, page 9 $\gamma = \frac{2}{49}$, $g = 9.8$, $m = 10$

(a) $\frac{dv}{dt} = 9.8 - \frac{2}{49(10)} v^2 = \frac{1}{245}\left((9.8)(245) - v^2\right) = \frac{1}{245}(49^2 - v^2)$

$\frac{dv}{dt} = \frac{1}{245}(49^2 - v^2)$ (This is a separable differential equation — see MAT 1102 for how to solve this — solution is below) See also partial fractions

(b) Suppose $v(0) = 0$

$\frac{1}{49^2 - v^2} = \frac{A}{49 - v} + \frac{B}{49 + v}$ or $1 = A(49 + v) + B(49 - v)$

Let $v = 49$ $\quad 1 = A(49 + 49) \Rightarrow A = \frac{1}{2(49)} = \frac{1}{98}$

Let $v = -49$ $\quad 1 = B(49 + 49) \Rightarrow B = \frac{1}{98}$

$\therefore \int \frac{1}{49^2 - v^2} dv = \int \frac{1}{245} dt$

$\frac{1}{98} \int \frac{1}{49 - v} dv + \frac{1}{98} \int \frac{1}{49 + v} dv = \frac{1}{245} t + C$

$\frac{1}{98}\left[-\ln|49 - v|\right] + \frac{1}{98} \ln|49 + v| = \frac{1}{245} t + C$

$\ln\left|\frac{49 + v}{49 - v}\right| = \frac{98}{245} t + C$

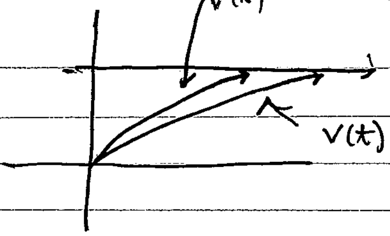
$\frac{49 + v}{49 - v} = K e^{\frac{98}{245} t}$

Since $v(0) = 0$ $\quad \frac{49 + 0}{49 - 0} = K(1) \Rightarrow K = 1$

$\frac{49 + v}{49 - v} = e^{\frac{98}{245} t} = e^{\frac{2(49)}{5(49)} t} = e^{.4t}$

$49 + v = (49 - v)e^{.4t} \Rightarrow v(1 + e^{.4t}) = 49e^{.4t} - 49$

$v(t) = 49\left[\frac{e^{.4t} - 1}{e^{.4t} + 1}\right]$ or $\boxed{v(t) = 49\left[\frac{1 - e^{-.4t}}{1 + e^{-.4t}}\right]}$

(c) 

(d) with quadratic drag force the terminal velocity is approached more rapidly.

(e) $$\frac{dx}{dt} = 49\left[\frac{1-e^{-.4t}}{1+e^{-.4t}}\right] = 49\left[1 - \frac{2e^{-.4t}}{1+e^{-.4t}}\right]$$

$$= 49\left[t + 5\ln(1+e^{-.4t})\right] + K$$

$$0 = x(0) = 49(5)\ln 2 + K \Rightarrow K = -49(5)\ln 2$$

$$x(t) = 49\left[t + 5\ln(1+e^{-.4t})\right] - 49(5)\ln 2$$

(f) $x(t) = 300$
Solution by calculator
$t = 9.9765321$ seconds