Method of Annihilators / (Sophistication of Undetermined Coefficients)

$D-a$ annihilates e^{at} because if $y=e^{at}$, then $y'-ay=0$

$(D-a)^2$ annihilates e^{at}, te^{at} because if $y=e^{at}$ or te^{at}
then $y''-2ay'+a^2y=0$

$(D-a)^n$ annihilates $e^{at}, te^{at}, \ldots t^{n-1}e^{at}$ because if $y=$ any of these,
then $(D-a)^n y(t)=0$.

(D^2+a^2) annihilates $\sin at$ and $\cos at$

$(D^2+a^2)^2$ annihilates $\sin at, \cos at, t\sin at, \cos at$

$(D^2+a^2)^n$ annihilates $t^j \sin at, t^j \cos at$ for $0 \le j \le n-1$

$D^2-2aD+a^2+b^2$ annihilates $e^{at}\cos bt, e^{at}\sin bt$

$(D^2-2aD+a^2+b^2)^n$ annihilates $t^j e^{at}\cos bt, t^j e^{at}\sin bt$

Read Method of Annihilators (starting bottom of page 237–238)
Here is how to do #6 page 237 using the technique at
bottom of page 238.

6. Find the general solution $y^{(4)}+2y''+y=3+\cos 2t$. $(*)$

$g(t)=3(1)+1\cos 2t$

$D=D-0$ annihilates $1=e^{0t}$

D^2+4 annihilates $\cos 2t$

(a) $\therefore H(D)=D(D^2+4)$ annihilates $3+\cos 2t$

$L(D)=D^4+2D^2+1=(D^2+1)^2$

$\therefore L(D)$ annihilates $c_1\sin t+c_2\cos t+c_3 t\sin t+c_4 t\cos t$

(b) $\therefore H(D)L(D)y=0$ has a general solution of

(c) $y(t)=c_1\sin t+c_2\cos t+c_3 t\sin t+c_4 t\cos t+c_5 e^{0t}+c_6\sin 2t+c_7\cos 2t$

(d) $y_p(t)=c_5+c_6\sin 2t+c_7\cos 2t$

$y_p'(t)= \quad 2c_6\cos 2t-2c_7\sin 2t$

$y_p''(t)= \quad -4c_6\sin 2t-4c_7\cos 2t$

$y_p'''(t)= \quad -8c_6\cos 2t+8c_7\sin 2t$

$y_p^{(4)}(t)= \quad 16c_6\sin 2t+16c_7\cos 2t$

$y_p^{(4)}(t)+2y_p''(t)+y_p(t)=c_5+9c_6\sin 2t+9c_7\cos 2t=3+\cos 2t$

$\therefore c_5=3 \quad c_6=0 \quad c_7=\frac{1}{9}\cos 2t$

$\therefore$ gen solution to $(*)$ is $\boxed{y(t)=3+\frac{1}{9}\cos 2t+c_1\sin t+c_2\cos t+c_3 t\sin t+c_4 t\cos t}$

#18 Page 237 using the method of annihilators

18. $y^{(4)} + 2y''' + 2y'' = 3e^{t} + 2te^{-t} + e^{-t}\sin t$

$D-1$ annihilates e^{t}

$(D+1)^2$ annihilates $2te^{-t}$

$D^2 + 2D + 1^2 + 1^2 = D^2 + 2D + 2$ annihilates $e^{-t}\sin t$

$\therefore H(D) = (D-1)(D+1)^2(D^2+2D+2)$ annihilates $3e^{t} + 2te^{-t} + e^{-t}\sin t$

$L(D) = D^4 + 2D^3 + 2D^2 = D^2(D^2+2D+2)$

$\therefore H(D)L(D) = (D-1)(D+1)^2(D^2+2D+2)^2 D^2$

$\therefore$ gen. solution to $H(D)L(D)y = 0$ is

$$y(t) = c_1e^{t} + c_2e^{-t} + c_3te^{-t} + c_4e^{-t}\sin t + c_5e^{-t}\cos t + c_6te^{-t}\sin t + c_7te^{-t}\cos t + c_8 + c_9t$$

terms that appear in solution to $L(D)y = 0$ are
D^2 terms $\quad c_8,\ c_9t$
$D^2+2D+2 \quad c_4e^{-t}\sin t,\ c_5e^{-t}\cos t$

$\therefore Y_p(t) = c_1e^{t} + c_2e^{-t} + c_3te^{-t} + c_6te^{-t}\sin t + c_7te^{-t}\cos t$

$\therefore$ gen solution is

$$y(t) = Y_p(t) + c_4e^{-t}\sin t + c_5e^{-t}\cos t + c_8 + c_9t$$

the problem does not ask us to find the specific values for $C_1, C_2, C_3, C_6,$ & C_7 in $Y_p(t)$.

Do #14, #16 using the method of annihilators.