Key to Test 1

(4) Prelim: $|x_n - L| = \left|\frac{5n-4}{7n+3} - \frac{5}{7}\right| = \frac{|35n - 28 - 35n - 15|}{|7(7n+3)|} = \frac{43}{7(7n+3)}$

$$\le \frac{43}{7(7n)} = \frac{43}{49n} \le \frac{43}{49}\frac{1}{N} < \epsilon \quad \text{or} \quad \frac{1}{N} < \frac{49}{43}\epsilon.$$

Given $\epsilon > 0$ choose $N \in \mathbb{N}$ (by the Arch. Property) so that $\frac{1}{N} < \frac{49}{43}\epsilon$.

$\therefore \forall n \ge N$, $\left|\frac{5n-4}{7n+3} - \frac{5}{7}\right| = \frac{43}{7(7n+3)} \le \frac{43}{49n} \le \frac{43}{49N} < \frac{43}{49}\frac{49}{43}\epsilon = \epsilon.$

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(5) Show that $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality

Suppose $n \in \mathbb{N}$. Then n is even or odd.
If n is even, then $\exists k \in \mathbb{N} \ni n = 2k$
If n is odd, then $\exists k \in \mathbb{N} \cup \{0\} \ni n = 2k+1$

Define $f(z) = \begin{cases} \frac{z}{2} & \text{if } z \text{ is even} \\ \frac{1-z}{2} & \text{if } z \text{ is odd} \end{cases}$

or $f(2k) = k \quad k \in \mathbb{N}$
$f(2k+1) = -k \quad k \in \mathbb{N} \cup \{0\}$.

$\therefore f: \mathbb{N} \to \mathbb{Z}$ and f is onto and f is 1-1 since
$k_1 = k_2 \Rightarrow 2k_1 = 2k_2$
$-k_1 = -k_2 \Rightarrow 2k_1 + 1 = 2k_2 + 1$

$\therefore f$ is a 1-1 correspondence from $\mathbb{N}$ to $\mathbb{Z}$
$\therefore \mathbb{N}$ and $\mathbb{Z}$ have the same cardinality.

(6) $a, b > 0$, $x_1 = a$, $x_{n+1} = \frac{.2}{b + x_n^{-1}} = \frac{.2x_n}{bx_n + 1} < \frac{.2}{1} x_n < x_n$

$\therefore x_n$ is decreasing and bounded below by zero
$\therefore$ By the monotone convergence theorem, x_n converges.

(7) $\frac{-1}{\sqrt{1+\frac{1}{n}}} \le x_n = \frac{n \sin(3n+5)}{\sqrt{n^2+n}} = \frac{n \sin(3n+5)}{n\sqrt{1+\frac{1}{n}}} = \frac{\sin(3n+5)}{\sqrt{1+\frac{1}{n}}} \le \sqrt{}$

$\therefore -\frac{1}{\sqrt{2}} \le x_n \le \frac{1}{\sqrt{2}}$

$\therefore x_n$ is a bounded sequence and has a convergent subsequence by the Bolzano Weierstrass theorem

⑧ Let $x_n = (4+(-1)^n 2)(1+\frac{(-1)^n}{n})$

then $x_{2k} = (4+2)(1+\frac{1}{2k}) = 6(1+\frac{1}{2k})$

$x_{2k+1} = (4-2)(1-\frac{1}{2k+1}) = 2(1-\frac{1}{2k+1})$

$\therefore \lim_{k\to+\infty} x_{2k} = 6$ and $\lim_{k\to+\infty} x_{2k+1} = 2$

The only possible limits for convergent subsequences of x_n are 2 and 4

$\therefore \liminf_{n\to+\infty} x_n = 2$ and $\limsup_{n\to+\infty} x_n = 6$

$\lim_{n\to+\infty} x_n$ does not exist since $\liminf_{n\to+\infty} x_n \neq \limsup_{n\to+\infty} x_n = 6$.

⑨ Given $M>0$ $\exists N\in\mathbb{N}$ so that $M<N$ (by Arch. Property)

$\therefore \forall n\geq N,\quad M<N\leq n\leq n(1+\frac{1}{n}) = x_n$.

$\therefore \lim_{n\to+\infty} x_n = +\infty$.

⑩ Let $\alpha>0$. By Arch. Property $\exists N \ni \frac{1}{N}<\alpha$

$\therefore \forall n\geq N,\quad 0<\frac{1}{\sqrt{2}n}\leq\frac{1}{\sqrt{2}N}<\frac{1}{N}<\alpha$.

$\frac{1}{\sqrt{2}n}$ is irrational for if rational then $\exists p,q\in\mathbb{N} \ni$ ($p\neq 0$)

$$\frac{1}{\sqrt{2}n} = \frac{p}{q} \Rightarrow \frac{1}{\sqrt{2}} = \frac{pn}{q} \Rightarrow \sqrt{2} = \frac{q}{pn}$$

and $\sqrt{2}$ would be rational. This is a contradiction.

$\therefore \frac{1}{\sqrt{2}n}$ is irrational.