Problem

① $|x_n - \frac{3}{4}| = \left|\frac{3n^2 + 8n\cos(\frac{\pi n}{8})}{4n^2+2} - \frac{3}{4}\right| = \left|\frac{\cancel{12n^2} + 32n\cos(\frac{\pi n}{8}) - \cancel{12n^2} - 6}{4(4n^2+2)}\right|$

$\leq \frac{32n+6}{4(4n^2+2)} = \frac{8(4n+\frac{6}{4})}{4(4n^2+2)} \leq \frac{2(4n+2)}{4n^2}$

$= \frac{2}{n} + \frac{1}{n^2} < \frac{3}{n}$

By Arch. Property $\exists N \in \mathbb{N} \ni \frac{1}{n} \leq \frac{1}{N} < \epsilon/3$.

$\therefore \forall n \geq N, \ |x_n - \frac{3}{4}| < \frac{3}{n} \leq \frac{3}{N} < 3(\epsilon/3) = \epsilon$.

Problem

②. $n = 16k$ $\quad x_n = \frac{3n^2 + 8n^2\cos(\frac{\pi n}{8})}{4n^2+2} = \frac{11n^2}{4n^2+2}$

$\therefore \lim_{k\to+\infty} \frac{11(16k)^2}{4(16k)^2+2} = \frac{11}{4} = \limsup x_n$

$n = 16k+8 \quad x_n = \frac{3n^2 - 8n^2}{4n^2+2} = \frac{-5n^2}{4n^2+2}$

$\lim_{k\to+\infty} \frac{-5n^2}{4n^2+2} = -\frac{5}{4} = \liminf x_n$.

$\limsup x_n \neq \liminf x_n \quad \therefore x_n$ does not converge

Problem

③. Assume $b_n > b_m \Rightarrow z_n = \int_a^{b_n} f > \int_a^{b_m} f = z_m$

$z_n - z_m = \int_a^{b_n} f - \int_a^{b_m} f = \int_{b_m}^{b_n} f \geq r(b_n - b_m)$

$\therefore |b_n - b_m| r \leq |z_n - z_m|$ or $|b_n - b_m| \leq \frac{1}{r}|z_n - z_m|$

z_n Cauchy $\Rightarrow$ Given $\epsilon > 0 \ \exists N \ni \forall n, m \geq N, \ |z_n - z_m| < r\epsilon$.

$\therefore \forall n, m \geq N, \ |b_n - b_m| \leq \frac{1}{r}|z_n - z_m| < \frac{1}{r} r\epsilon = \epsilon$.

$\therefore b_n$ is Cauchy $\Rightarrow b_n$ converges.

Problem ③ $a_0 = 0$, $a_1 = 1$ $a_n = \frac{a_{n-1} + a_{n-2}}{2}$ for $n \geq 2$.

① $\therefore$ $a_k \in [0,1]$ $\forall k \in \mathbb{N}$.

② a_k has a convergent subsequence By BW whose limit is an element of $[0,1]$.

③

	$a_k - a_{k-1}$
$a_0 = 0$	
$a_1 = 1$	1
$a_2 = \frac{1}{2}$	$-\frac{1}{2}$
$a_3 = \frac{3}{4}$	$\frac{1}{4}$
$a_4 = \frac{5}{8}$	$-\frac{1}{8}$
$a_5 = \frac{11}{16}$	$\frac{1}{16}$
$a_6 = \frac{21}{32}$	$-\frac{1}{32}$

$\therefore a_k = \sum_{j=1}^{k} (-1)^{j-1} \frac{1}{2^{j-1}}$

$\therefore \lim_{k \to \infty} a_k = \frac{1}{1 - (-\frac{1}{2})} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$

Problem ⑤ g is bounded on $\mathbb{R}$. $\exists M > 0 \ni \forall x \in \mathbb{R}$ $g(x) \in [-M, M]$

Since f' is cont. on $\mathbb{R}$,

f' on $[-M, M]$ is bounded

$\therefore \exists K_1 > 0 \ni \forall x \in \mathbb{R}$, $|f'(g(x))| \leq K_1$.

Let K_2 be the bound on the derivative of g

By MVT, $\forall x, y \in \mathbb{R}$ $\exists c$ between x & y so that

$$\left| \frac{f(g(x)) - f(g(y))}{x-y} \right| = |f'(g(c)) g'(c)| \leq K_1 K_2$$

$\therefore |f(g(x)) - f(g(y))| \leq K_1 K_2 |x-y|$

Given $\epsilon > 0$ choose $\delta > 0 = \epsilon / K_1 K_2$. then $|f(g(x)) - f(g(y))| \leq K_1 K_2 |x-y| < \epsilon$ whenever $|x-y| < \delta$.

Problem

6. By the Extreme Value Theorem since f & g are cont.
$\exists\, c_1, c_2 \in [a,b]$ so that $\forall x \in [a,b]$
$$f(c_1) \le f(x) \le h(x) \le g(x) \le g(c_2)$$
$\therefore$ ~~the~~ h is bounded below by $f(c_1)$
and above by $g(c_2)$.

By 41 (a) & (b), page 207, the upper & lower Darboux integrals exist.

(b) (i) $\underline{\int_0^2} f = (.5)(2) = 1$

(ii) $\overline{\int_0^2} f = .75(2) = 1.5$