Solutions - Assignment from Pages 8-9

(6) Prove that the reciprocal of an irrational number is irrational.

Proof: Let x be an irrational number.
Suppose $\frac{1}{x}$ is rational. Then $\exists\, p,q \in \mathbb{Z}$, $q \neq 0$ + $\frac{1}{x} = \frac{p}{q}$
Neither p nor q are zero because $q \neq 0$ already and if $p = 0$ then $\frac{1}{x} = 0$ and the multiplicative inverse of 0 would be x (but 0 has no multiplicative inverse)

$\therefore x = \frac{q}{p}$ $p \neq 0$. $\therefore$ x is rational.

This is a contradiction to x being irrational.

$\therefore \frac{1}{x}$ must be irrational.

(7) Let $x \in \mathbb{Q}$ and y be irrational.
Suppose $x+y$ is rational

$\therefore \exists\, p_1, q_1, p_2, q_2 \in \mathbb{Z}$, $q_1 \neq 0, q_2 \neq 0$ +

$$\frac{p_1}{q_1} + y = \frac{p_2}{q_2}$$

$$\therefore y = \frac{p_2}{q_2} - \frac{p_1}{q_1} = \frac{p_2 q_1 - p_1 q_2}{q_1 q_2}$$

$\therefore y$ is rational.
This is a contradiction
$\therefore x+y$ must be irrational.

(10) Let x be a real number, then either $\sqrt{2} - x$ or $\sqrt{2} + x$ is irrational.

Proof: Suppose $\sqrt{2} - x$ and $\sqrt{2} + x$ are both rational

Then $\exists\, p, q \in \mathbb{Z}$, $q \neq 0$ so that $\sqrt{2} - x + (\sqrt{2} + x) = \frac{p}{q}$

or $2\sqrt{2} = \frac{p}{q}$ or $\sqrt{2} = \frac{p}{2q}$

$\therefore \sqrt{2}$ is rational - This is false
$\therefore$ one of $\sqrt{2} - x$ or $\sqrt{2} + x$ is rational

(6) $\emptyset \neq S \subseteq A$ and A is bounded

Outline of Solutions Assignment from Page 29 of Text

(11)

$\exists M > 0 \ni \forall x \in A, \ |x| \leq M$

$\therefore \forall x \in A, \quad -M \leq x \leq M$

$\therefore$ sup(A) exists and $\sup A \leq M$

$\therefore$ inf(A) exists $\quad -M \leq \inf A$

Since $S \subseteq A$, $\forall x \in S$, $x \in A$, $\inf A \leq x \leq \sup A$

$\therefore$ $\sup S \leq \sup A$ (3)

and $\inf A \leq \inf S$ (1)

and $\inf S \leq \sup S$ (2)

$\therefore \quad \inf A \underset{(1)}{\leq} \inf S \underset{(2)}{\leq} \sup S \underset{(3)}{\leq} \sup A$

(27) $A \neq \emptyset$, A is bounded

$\therefore$ inf A & sup A exists.

<u>$\exists x \in A$</u> $\ni$ * $\forall a \in A$, $a \leq x$. (x is called the maximum of A)

~~$\therefore$ x is an upper bound for A~~

~~$\therefore$ sup A $\leq$ x~~

~~$\therefore$ x = sup A~~

($\Rightarrow$) Suppose A has a maximum x.

$\forall a \in A$, $a \leq x$

$\therefore$ x is an upper bound for A

sup A $\leq$ x $\in A$

Suppose sup A $\nleq$ x

~~$\exists \dots \in A \ni \dots$ sup A $\leq \dots <$ x~~

This contradicts *

and $\therefore$ x $\leq$ sup A

$\therefore$ x = sup A

$\therefore$ A contains its supremum because x is in A

($\Leftarrow$) A contains its sup.

$\therefore$ sup A $\in$ A

$\therefore$ <u>$\forall a \in A$, $a \leq$ sup A</u> and sup A $\in$ A

$\therefore$ sup A = max A

(28) $A \subseteq A \cup B$ (13)

$B \subseteq A \cup B$

$\inf(A \cup B) \le \inf A \le \sup A \le \sup(A \cup B)$

$\inf(A \cup B) \le \inf B \le \sup B \le \sup(A \cup B)$

① $\inf(A \cup B) \le \min\{\inf A, \inf B\}$

② $\max\{\sup A, \sup B\} \le \sup(A \cup B)$

Let $x \in A \cup B$

$\therefore$ $x \in A$ or $x \in B$

$\inf A \le x$ or $\inf B \le x$

$\therefore$ $\min\{\inf A, \inf B\} \le x$

$\min\{\inf A, \inf B\}$ is a lower bound for $A \cup B$

(1') $\min\{\inf A, \inf B\} \le \inf(A \cup B)$

$\therefore$ $\inf(A \cup B) = \min\{\inf A, \inf B\}$

Let $x \in A \cup B$

$x \in A$ or $x \in B$

$x \le \sup A$ or $x \le \sup B$

$\therefore$ $x \le \max\{\sup A, \sup B\}$

$\therefore$ $\max\{\sup A, \sup B\}$ is an upper bound for $A \cup B$

(2') $\sup\{A \cup B\} \le \max\{\sup A, \sup B\}$

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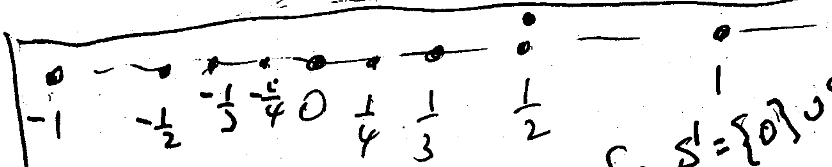
Example $S = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{-\frac{1}{2}\}$

$-S = \{0\} \cup \{\frac{1}{2}\} \cup \{-\frac{1}{n} : n \in \mathbb{N}\}$



for $S' = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$

$\sup(-S) = 0$

$-\sup(-S) = -0 = 0 = \inf S$

$\sup(-S) = \frac{1}{2}$ for S'

$-\sup(-S) = -\frac{1}{2}$

$\inf S' = -\frac{1}{2}$

$\inf S = -\sup(-S)$

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(29) S ~~is~~ has $\alpha = \inf S$ and $\beta = \sup S$

$\forall x \in S$, ① $\alpha \le x$ and no number a larger than α is an ~~upper~~ lower bound

if $\alpha < a$, there is an $x \in S \ni$ $\alpha < x \le a$.

$\forall x \in S$, $-\alpha \ge -x$

$\forall y \in -S$, $-\alpha \ge y \Rightarrow$ ① $-\alpha$ is an upper bound for $-S$

$\therefore -\inf S$ is an upper bound for $-S$

if $-\alpha > -a$, there is an $x \in S \ni$

$$-\alpha \geq -x > -a$$

$-a$ is not an upper bound for $-S$

$\therefore -\alpha = \sup(-S)$

$\alpha = -\sup(-S)$

$\inf S = -\sup(-S)$

Section 1.4 — outline

one-to-one correspondence
cardinality

finite $\emptyset$ or $\{1, 2, \ldots, n\}$
infinite – not finite
countably infinite – $\mathbb{N}$
countable finite or countably infinite
uncountable – not countable

$\mathbb{N}$ ~~Z~~ $\mathbb{N} \cup \{0\}$ $\mathbb{Z}$, $\mathbb{Q}$

$\mathbb{R}$ is uncountable

The set of irrationals are uncountable