natural numbers $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

integers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

rational numbers $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$

rational repeating or terminating decimals

$$\begin{array}{r} .42857\,1 \\ 7\,\overline{)\,3.0} \\ \underline{28} \\ 20 \\ \underline{14} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 50 \\ \underline{49} \\ 10 \\ \underline{7} \\ 3 \end{array}$$

$\frac{3}{7} = .\overline{428571}$

$x = .\overline{42857}$

$100000x = 42857.\overline{42857}$

$x = .\overline{42857}$

$99999x = 42857$

$x = \frac{42857}{99999}$

$x = .420\overline{703}$

$10^6 x = 420703.\overline{703}$

$10^3 x = 420.\overline{703}$

$999000x = 420283$

$x = \frac{420283}{999000}$

$.101001000100001000001 \ldots$ neither terminating nor repeating

1 3 6 10

irrational = numbers that are not rational

Let $p \in \mathbb{N}$

Thm If $2 \mid p^2$, then $2 \mid p$.

Proof: Assume $2 \nmid p$

$p = 2k+1$

$p^2 = 4k^2 + 4k + 1$

$p^2 = 2(2k^2 + 2k) + 1$

$\therefore 2 \nmid p^2$.

Parallel Let q be prime and $p \in \mathbb{N}$

Thm If $q \mid p^2$ then $q \mid p$.

Proof: Assume $q \nmid p$

$p = qk + n \quad 0 < n < q - 1$

$p^2 = q^2k^2 + 2qkn + n^2$

$= q(qk^2 + 2kn) + n^2$

$\therefore q \mid p^2 \Rightarrow q \mid n$

$n = n_1 n_2 \cdots n_l$ prime factorization

$n^2 = n_1^2 n_2^2 \cdots n_l^2$

q not in prime factorization of n^2.

$\therefore q \nmid n^2$.

$\sqrt{2}$ is irrational

Assume $\sqrt{2}$ is rational

$\exists p, q \in \mathbb{N}$ no common factors

$\sqrt{2} = \frac{p}{q}$

$2 = \frac{p^2}{q^2}$

$p^2 = 2q^2$

$2 \mid p^2 \Rightarrow 2 \mid p \Rightarrow p = 2k$

$p^2 = \cancel{\ldots}\ 4k^2 = 2q^2$

$\therefore \cancel{2 \nmid p^2}\ q^2 = 2k^2$

$2 \mid q^2 \Rightarrow 2 \mid q$

$\therefore p$ & q have a common factor

j prime, then $\sqrt{j}$ is irrational

Assume $\sqrt{j}$ is rational

$\sqrt{j} = \frac{p}{q}$

$j = \frac{p^2}{q^2}$

$p^2 = jq^2$

$j \mid p^2 \Rightarrow j \mid p \Rightarrow p = jk$

$p^2 = \cancel{\ldots}$

$j^2k^2 = jq^2$

$jk^2 = q^2$

$\therefore j \mid q^2 \Rightarrow j \mid q$

but p & q have no common factors

Major themes of 1.1 1.2 1.3

1.1 $\mathbb{R}$ is an ordered field

1.2 (more about the order relation $<$)

1.3 $\mathbb{R}$ is a complete ordered field.

Consequence: thm 1.20 Every real number has a decimal expansion.

field nonemptyset F, two operations $+, *$

closure	$\forall x, y \in F$	① $x+y \in F$	⑥ $xy \in F$
comm.		② $x+y = y+x$	⑦ $xy = yx$
assoc.		③ $(x+y)+z = x+(y+z)$	⑧ $(xy)z = x(yz)$
identity	$\forall x \in F$	④ $x+0 = x$	⑨ $x1 = x$
inverse		⑤ $\forall x \in F\ \exists y \in F \ni x+y=0$	⑩ $\forall x \neq 0, \exists y \in F \ni xy = 1$
distributive	$\forall x, y, z \in F$	⑪ $x(y+z) = xy + xz$	

Mention Thomas Harriott – wiki article

Known for • Introducing symbols for "is less than" ($<$) and "is greater than" $>$

• Translation of the Carolina Algonquian language into English.

Def: relation $<$ on a set S $\ni$

① $\forall x, y \in S$, then exactly one is true $x<y$, $x=y$, or $y<x$ — trichometry

② $\forall x, y, z \in S$, if $x<y$ and $y<z$, then $x<z$ — transitivity

ordered set

Def ordered field — field with order $<$ plus

① If $0<x$ and $0<y$, then $0<x+y$

② If $0<x$ and $0<y$, then $0<xy$

③ $x<y$ if and only if $0<y-x$.

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Homework:

3c — #12 Wednesday

4c — #17 Friday

6 Monday

7 Monday

8

10 Monday

20

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

1.4 $\quad -|a| \le a \le |a|$

$|ab| = |a||b|$

$|-a| = |a|$

Let $c > 0$ $\quad |a| < c$ iff $-c < a < c$

$|a| \le c$ iff $-c \le a \le c$

Triangle Inequality $\quad |a+b| \le |a| + |b|$

Reverse " " $\quad ||a| - |b|| \le |a - b|$

Theorem 1.7 Let x be a real number. If $|x| < \epsilon$ $\forall \epsilon > 0$, then $x = 0$.

Proof: By contradiction. Let $x \neq 0$.

$\therefore \frac{|x|}{2} > 0$. Let $\frac{|x|}{2} = \epsilon$.

$\therefore |x| < \frac{|x|}{2}$

$\#$

$\therefore x = 0$.

$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2$

$a + ar + \cdots + ar^n = a \frac{1-r^{n+1}}{1-r}$

(31) $\quad 0 \le P(t) = \sum_{k=1}^{n} (a_k - t b_k)^2 = \sum_{k=1}^{n} a_k^2 - 2t \sum_{k=1}^{n} a_k b_k + t^2 \sum b_k^2$

$P'(t) = -2 \sum_{k=1}^{n} a_k b_k + 2t \sum b_k^2 = 0$

$t = \frac{\sum_{k=1}^{n} a_k b_k}{\sum b_k^2}$

$0 \le \sum a_k^2 - 2 \frac{\left(\sum_{k=1}^{n} a_k b_k\right)^2}{\sum b_k^2} + \frac{\left(\sum a_k b_k\right)^2}{\sum b_k^2}$

- bounded above
  upper bound
- bounded below
  lower bound
- bounded
  bound
- ~~and~~ unbounded

Examples

Def: Let $S \subseteq \mathbb{R}$, $S \neq \emptyset$.

a) Suppose S is bounded above

$\beta = \sup S$ if β is an upper bound for S
supremum and if $x < \beta$, then x is not an upper bound for S
(if x is ~~and~~ upper bound for S, then $x \geq \beta$)
$\therefore$ β is the least upper bound for S

(b) Suppose S is bounded below
$\alpha = \inf S$ if α is a lower bound for S
and if $\alpha < x$, then x is not a lower bound for S.

see text argument that $S = \{x \in \mathbb{Q} : x > 0 ~~\&~~ + x^2 < 2\}$
is nonempty bounded but has no sup among rationals ~~sup above~~

~~(?)~~ Completeness Axiom: Each nonempty set of real numbers that ~~has~~ is bounded above has a supremum.

Theorem 1.6 Arch Property

(1) If $a, b > 0$, $\exists$ $n \in \mathbb{N}$ $\ni$ $na > b$

Proof: $a \geq b$ take $n = 2$
consider $0 < a < b$ and $\forall n$ $na \leq b$
$\therefore$ $S = \{na : n \in \mathbb{N}\}$ bounded above by b
$\therefore$ $\beta = \sup S$ exists
$\beta - a < \beta$
$\therefore$ $\beta - a$ is not an upper bound
$\therefore$ $\exists$ $p \in \mathbb{N}$ $\ni$ $pa > \beta - a$
$\therefore$ $(p+1)a > \beta$
but $(p+1)a \in S$
$\therefore$ β not an upper bound

Theorem 1.17 TFAE

1. $a, b > 0$. $\exists\, n \in \mathbb{N} \ni na > b$
2. $\mathbb{N}$ is not bounded above
3. $\forall\, x \in \mathbb{R} \quad \exists\, n \in \mathbb{Z} \ni n \le x < n+1$
4. $\forall\, x > 0, \quad \exists\, n \in \mathbb{N} \ni \frac{1}{n} < x.$

Proof $(2) \Rightarrow (3)$
$(4) \Rightarrow 1$ in book

$(1') \Rightarrow (2)$
$(3) \Rightarrow (4)$ exercises

$(1) \Rightarrow (2)$ Suppose $\mathbb{N}$ bounded above by b

Take $a = 1$

$\exists\, n \in \mathbb{N} \quad n \cdot 1 > b$

$\therefore$ b not an upper bound

$(3) \Rightarrow (4)$ Let $x > 0$ By (3) $\exists\, n \in \mathbb{Z} \ni n \le \frac{1}{x} < n+1$

$\therefore \quad 0 < \frac{1}{x} < n+1$

$\therefore \quad n+1 \in \mathbb{N}$ and $\frac{1}{n+1} < x$.

Thm 1.18 Let $x < y$ $\exists\, p, q \in \mathbb{N} \ni x < \frac{p}{q} < y$

Proof: $x < y \Rightarrow 0 < y - x$

$\exists\, q \in \mathbb{N} \ni q(y-x) > 1$

or $1 < qy - qx$

$\exists\, p \in \mathbb{Z} \ni qx < p < qy \Rightarrow x < \frac{p}{q} < y$

Apply same argument to $\sqrt{2}x < \sqrt{2}y \Rightarrow \sqrt{2}x < \frac{p}{q} < \sqrt{2}y$

$x < \frac{p}{\sqrt{2}q} < y$.

Also irrational

Theorem 1.20 Each positive real number has a decimal expansion.

Proof: Let $x > 0$.
By Arch $\exists\, d_0 \in \mathbb{Z} \ni d_0 \le x < d_0 + 1$

Let $d_1 \in \mathbb{Z} \ni d_1$ is the largest integer $\ni d_0 + \frac{d_1}{10} \le x$

(Note: $0 \le d_1 \le 9$ for if $10 \le d_1$, then $d_0 + \frac{d_1}{10} \ge d_0 + 1 > x$)

Define d_2 largest integer $\ni d_0 + \frac{d_1}{10} + \frac{d_2}{10^2} \le x$

(Note $0 \le d_2 \le 9$, if $d_2 \ge 10$, then $d_0 + \frac{d_1}{10} + \frac{d_2}{10^2} \ge d_0 + \frac{d_1}{10} + \frac{1}{10} > x$)

Define $d_n \in \{0, 1, \ldots, 9\} \ni$

$$d_0 + \frac{d_1}{10} + \cdots + \frac{d_n}{10^n} \le x \quad \text{and} \quad d_0 + \frac{d_1}{10} + \cdots + \frac{d_n + 1}{10^n} > x$$

Let $D = \{d_0 + \frac{d_1}{10} + \cdots + \frac{d_n}{10^n} : n \in \mathbb{N} \cup \{0\}\} \ne \emptyset$
and D has an upper bound of x.

$\therefore \beta = \sup D$ exists and $\beta \le x$

Now we will show that $\beta = x$.
Assume to the contrary that $\beta < x$
By Arch $\exists\, p \in \mathbb{N} \ni \frac{1}{p} < x - \beta$.

$\therefore x < d_0 + \frac{d_1}{10} + \cdots + \frac{d_p}{10^p} + \frac{1}{10^p} \le \beta + \frac{1}{10^p} < \beta + \frac{1}{p} < x$

this is a contradiction. $\therefore \beta = x$

$\therefore d_0 . d_1 \ldots d_p \le x = \beta < d_0 . d_1 \ldots d_p + \frac{1}{10^p}$

$\therefore x = d_0 . d_1 d_2 \ldots$

Section 1.3 Page 27 (Some easy proofs)

(2) Prove that the set $S = \{x \in \mathbb{R} : 10\sqrt{x} - x > 0\}$ is bounded

Note that $S \subseteq [0, +\infty)$ since $x \in S$ for $\sqrt{x}$ to be defined

$\therefore$ 0 is a lower bound for S

Suppose $x \in S$, then

$$\begin{aligned} 10\sqrt{x} - x &> 0 \\ 10\sqrt{x} &> x \\ 10 &> \sqrt{x} \\ 100 &> x \end{aligned}$$

$\therefore$ 100 is an upper bound for S

$\therefore S$ is bounded because $\forall x \in S$, $x = |x| < 100$.

(6) Prove that the union of two bounded sets is a bounded set.

Proof: Suppose A and B are bounded.

$\therefore$ ~~$\forall x$~~ $\exists M_1 > 0 \ni \forall x \in A$, $|x| \leq M_1$

$\exists M_2 > 0 \ni \forall x \in B$, $|x| \leq M_2$

Let $M = \max\{M_1, M_2\}$.

Let $x \in A \cup B$. if $x \in A$ then $|x| \leq M_1 \leq M$

if $x \in B$ then $|x| \leq M_2 \leq M$.

$\therefore \forall x \in A \cup B$, $|x| < M$.

(8) Prove that a nonempty set that is bounded above has only one supremum.

Proof: Let $S \neq \emptyset$ and $\exists M > 0$ $\forall x \in S$, $|x| < M$.

By the Completeness Axiom $\exists\, \beta_1 = \sup M$

Suppose $\exists\, \beta_2 = \sup M \ni \beta_1 < \beta_2$.

$\therefore$ By definition β_1 is not an upper bound for S.

This is a contradiction. $\therefore \beta_2 \leq \beta_1$. A similar argument shows $\beta_1 \leq \beta_2$

$\therefore \beta_1 = \beta_2$.

(12) Let S be a nonempty set of real numbers that is bounded above and let $\beta = \sup S$. Prove that for each $\epsilon > 0$ there exists a point $x \in S \ni x > \beta - \epsilon$.

Proof: Let $S \neq \emptyset$ and that S is bounded above and $\beta = \sup S$.
Suppose the conclusion is false,
i.e. $\exists \epsilon > 0 \ni \forall x \in S, \quad x \leq \beta - \epsilon$.

Then $\beta - \epsilon$ is an upper bound for S
and $\beta = \sup S \leq \beta - \epsilon \Rightarrow \quad 0 < -\epsilon$
or $\epsilon < 0$.
this is a contradiction to $\epsilon > 0$.
$\therefore$ the conclusion is true.

Homework for Friday August 31.

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Homework for Wednesday, September 5

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Page 35 Problems 3, 4, 17