Test 3 Spring, 2015 Name: Key
MAT 162 version 1 Gurganus

Directions: Show all work for partial credit purposes. You may use a graphing calculator. The test is closed book.

1. Use Euler's Method to approximate y(1.2) given $\frac{dy}{dx} = 10x+y^2$, and y(1) = 2 Use a stepsize of 0.1 .

(16)

2 for each position

| x | y | $(10x+y^2).1$ |
| --- | --- | --- |
| 1 | 2 | $(14)(.1) = 1.4$ |
| 1.1 | 3.4 | $(11+(3.4)^2).1 = 2.256$ |
| 1.2 | $\boxed{5.656}$ | |

2. Find y(x), the solution to $\frac{dy}{dx} = (y^2+1)(x^3+2x)$ , y(0) = π/4 .

(16)

3 $\frac{1}{y^2+1}\,dy = (x^3+2x)\,dx$

3 $\arctan y = \frac{1}{4}x^4 + x^2 + C$

3 $\arctan\frac{\pi}{4} = C$

4 $\arctan y = \frac{1}{4}x^4 + x^2 + \arctan\frac{\pi}{4}$

3 $y = \tan\left(\frac{1}{4}x^4 + x^2 + \arctan\frac{\pi}{4}\right)$

$\arctan\frac{\pi}{4} = .66577536$

3. Find y(x), the solution to $\frac{dy}{dx} = xe^{-\sin(x)} - y\cos(x)$ , y(0) = 2 .

(16)

2 $\frac{dy}{dx} + \cos x\, y = xe^{-\sin x}$

2 $I(x) = e^{\int \cos x\,dx} = e^{\sin x}$

2 $\frac{d}{dx}\left(e^{\sin x}y\right) = x$

2 $e^{\sin x}y = \frac{1}{2}x^2 + C$

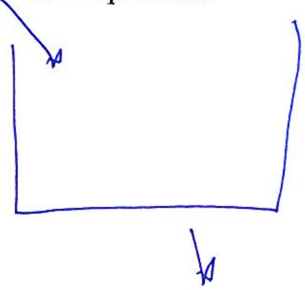
2 $e^0(2) = 0 + C$

$2 = C$

2 $e^{\sin x}y = \frac{1}{2}x^2 + 2$

2 $y = e^{-\sin x}\left(\frac{1}{2}x^2 + 2\right)$

4. A tank is filled with 1000 liters of contaminated water containing 50 g of salt. Water containing .1 g of salt per liter is pumped in at a rate of 10 l/min., mixes instantaneously, and then is pumped out at a rate 0f 15 l/min.. Find y(t) the number of grams of the salt in the tank t minutes after the rinse begins. Then find the time at which there is 10 g of ~~toxin~~ salt present.



$s(t) =$ g of salt at time $t$

$s(0) = 50$

$\frac{ds}{dt} =$ rate in $-$ rate out

$= .1\, g/l \;\; 10\, l/min - \frac{s(t)}{1000-5t}(15)$

$= 1 - \frac{3}{200-t}\, s(t)$

$\frac{ds}{dt} + \frac{3}{200-t}\, s(t) = 1$

$I(t) = e^{\int \frac{3}{200-t}\,dt} = e^{-3\ln(200-t)} = (200-t)^{-3}$

$\frac{d}{dt}\left((200-t)^{-3} s\right) = (200-t)^{-3}$

$(200-t)^{-3} s(t) = \frac{\cancel{200}\,(200-t)^{-2}}{2} + C$

$200^{-3}(50) = \frac{(200)^{-2}}{2} + C \Rightarrow C = -6.25\ 10^{-4}$

$(200)^{-3}(100) = 200^{-2} + 2C$

$(200)^{-3}(100) - 200^{-2} = 2C = 8^{-3}\left[10^{-4}\right] - 2^{-2}\left[10^{-4}\right] = 10^{-4}\left[\frac{1}{8^3} - \frac{1}{2^2}\right]$

$\cancel{s(t) = \frac{(200-t)}{2}} + 10^{-4}\left[\frac{1}{8^3} - \frac{1}{2^2}\right]\frac{1}{2}\,(\cancel{200}-t)$

Find $s(t^*) = 10$

$s(t^*) = 10$

$t = 179.9$ min.

5. First find the solution to $\frac{d^2y}{dx^2} + 22\frac{dy}{dx} - 75y = 0$, $y(0) = 1$, $y'(0) = 2$.

$r^2 + 22r - 75 = 0$

$(r+25)(r-3)$

$y(t) = c_1 e^{-25t} + c_2 e^{3t}$

$y'(t) = -25 c_1 e^{-25t} + 3 c_2 e^{3t}$

$1 = y(0) = c_1 + c_2$

$2 = y'(0) = -25c_1 + 3c_2$

$25 = 25c_1 + 25c_2$

$2 = -25c_1 + 3c_2$

$27 = 28 c_2$

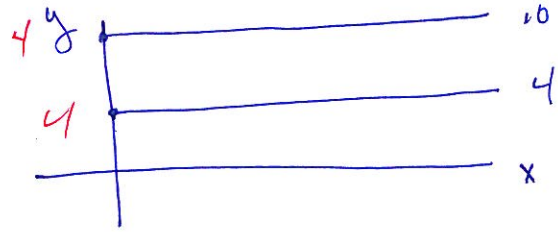
$\frac{27}{28} = c_2$

$c_1 = 1 - c_2 = 1 - \frac{27}{28} = \frac{1}{28}$

6. For $\frac{dy}{dx}$ =-2 (y -4)( y-10)

a. Sketch the equilibrium solutions.

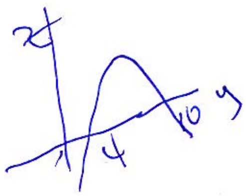
(8|

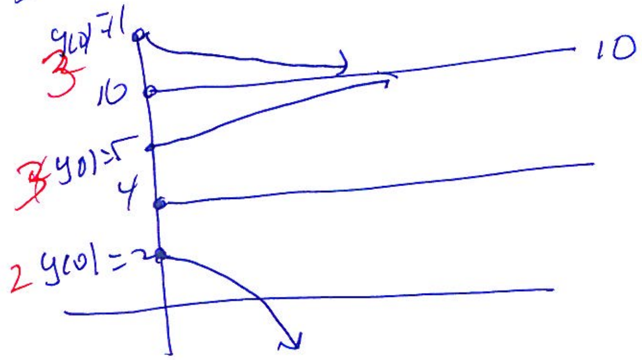



b. Sketch the solutions for the initial conditions y(0) = 2, y(0) = 5, and y(0) = 11.

$y' = -2(y-4)(y-10)$

(8 3




c. (10 points extra credit) Solve $\frac{dy}{dx}$ =-2 (y -3)( y-9) , y(0) = 5.

4 $\int \frac{1}{(y-3)(y-9)}\,dy = \int -2\,dx \quad y(0)=5$

$\frac{1}{(y-3)(y-9)} = \frac{A}{y-3} + \frac{B}{y-9}$

$1 = A(y-9) + B(y-3)$

$1 = A(3-9) \Rightarrow A = -\frac{1}{6}$

$1 = B(6) \Rightarrow B = \frac{1}{6}$

2 $\int \frac{1}{6}\left[\frac{1}{y-9} - \frac{1}{y-3}\right] dy = \int -2\,dx$

$\frac{1}{6}\ln\left|\frac{y-9}{y-3}\right| = -2x + C$

$\ln\left|\frac{y-9}{y-3}\right| = -12x + C$

2

$\frac{y-9}{y-3} = Ke^{-12x}$

$-2 = \frac{-4}{2} = Ke^{-12\cdot 0}$

2

$\frac{y-9}{y-3} = -2e^{-12x}$

$\frac{y-9}{y-3} = w$

$y-9 = wy - 3w$

$y(1-w) = 9-3w$

$y = \frac{9-3w}{1-w}$

$\boxed{y = \frac{9+6e^{-12x}}{1+2e^{-12x}}}$

Test 3 Spring, 2015 Name: Key

MAT 162 version 2 Gurganus

Directions: Show all work for partial credit purposes. You may use a graphing calculator. The test is closed book.

1. Use Euler's Method to approximate y(1.2) given $\frac{dy}{dx} = 10y+x^2$, and y(1) = 2 Use a stepsize of 0.1 .

16 ~~8~~ (2 each)

| x | y | $(10y+x^2)(.1)$ |
| --- | --- | --- |
| 1 | 2 | $(21)(.1) = \cancel{1.4}\ 2.1$ |
| 1.1 | ~~3.4~~ 4.1 | $(\cancel{34}\ 41 + 1.21)(.1) = \cancel{3.521}$ Answer 4.221 |
| 1.2 | ~~8.521~~ ~~8.741~~ 8.321 | |

2. Find y(x), the solution to $\frac{dy}{dx} = (y^2 + 1)(x^3 + 2x)$ , y(0) = 1 .

16

4 $\frac{1}{y^2+1}dy = (x^3+2x)dx$

4 $\arctan y = \frac{1}{4}x^4 + x^2 + C$

4 $\frac{\pi}{4} = \arctan 1 = 0 + C$

4 $\boxed{y = \tan\left(\frac{1}{4}x^4 + x^2 + \frac{\pi}{4}\right)}$

3. Find y(x), the solution to $\frac{dy}{dx} = xe^{-\sin(x)} - y\cos(x)$ , y(0) = 2 .

16

3 $\frac{dy}{dx} + \cos x\, y = xe^{-\sin x}$

3 $I(x) = e^{\int \cos x\, dx} = e^{\sin x}$

3 $\frac{d}{dx}(e^{\sin x}y) = x$

$e^{\sin x}y = \frac{1}{2}x^2 + C$

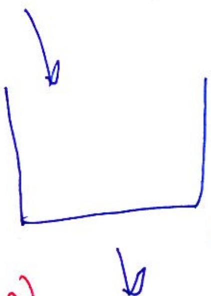
3

2 $e^{\sin(0)}2 = \frac{1}{2}(0^2) + C$

$2 = C$

2 $y = e^{-\sin x}\left(\frac{1}{2}x^2 + 2\right)$

4. A tank is filled with 1000 liters of contaminated water containing 50 g of salt. Water containing .1 g of salt per liter is pumped in at a rate of 10 l/min., mixes instantaneously, and then is pumped out at a rate 0f 15 l/min.. Find y(t) the number of grams of the salt in the tank t minutes after the rinse begins. Then find the time at which there is 20 g of salt present.



2 $S(t)$ = grams of salt at t minutes

2 $S(0) = 50$

4 $\frac{ds}{dt}$ = rate in − rate out $= (.1)\, g/l \; 10 l/min - \frac{S}{1000-5t}(15)$

$= 1 - \frac{3}{200-t} S$

(20)

2 $\frac{ds}{dt} + \frac{3}{200-t} S = 1$

$I(t) = e^{\int \frac{3}{200-t} dt} = e^{-3 \ln(200-t)} = \frac{1}{(200-t)^3}$

2 $\frac{d}{dt}\left(\frac{1}{(200-t)^3} S(t)\right) = 1(200-t)^{-3}$

2 $\frac{1}{(200-t)^3} S(t) = \frac{(200-t)^{-2}}{2} + C$

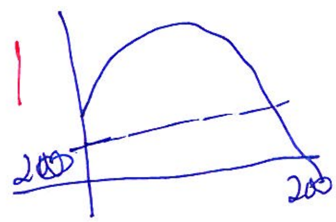
2 $\frac{1}{(200)^3} 50 = \frac{1}{2(200)^2} + C$

1 $C = \frac{1}{(200)^3} 50 - \frac{1}{2(200)^2} = \frac{-50}{(200)^3}$

$S(t) = \frac{1}{2}(200-t) - \frac{50}{200^3}(200-t)^3$

Solve for $S(t) = 20$

1




$t = 159.14777$ minutes.

2 5. First find the solution to $\frac{d^2y}{dx^2}$ -22$\frac{dy}{dx}$ -75y = 0 , y(0) = 1, y'(0) = 2.

16

2 $r^2 - 22r - 75 = 0$

2 $(r+3)(r-25) = 0$

$r = -3 \quad r = 25$

2 $y(x) = c_1 e^{25x} + c_2 e^{-3x}$

2 $y'(x) = 25 c_1 e^{25x} - 3c_2 e^{-3x}$

4 $\begin{cases} 1 = y(0) = c_1 + c_2 \\ 2 = 25c_1 - 3c_2 \end{cases}$

2 $\begin{cases} 3 = 3c_1 + 3c_2 \\ 2 = 25c_1 - 3c_2 \end{cases}$

$5 = 28c_1$

$c_1 = \frac{5}{28}$

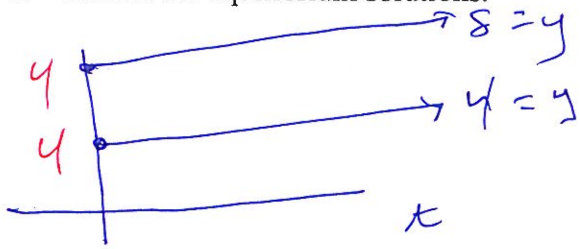
$c_2 = 1 - c_1 = \frac{23}{28}$

2 $\boxed{y(x) = \frac{5}{28} e^{25x} + \frac{23}{28} e^{-3x}}$

6. For $\frac{dy}{dx}$ =-2 (y -4)( y-8)

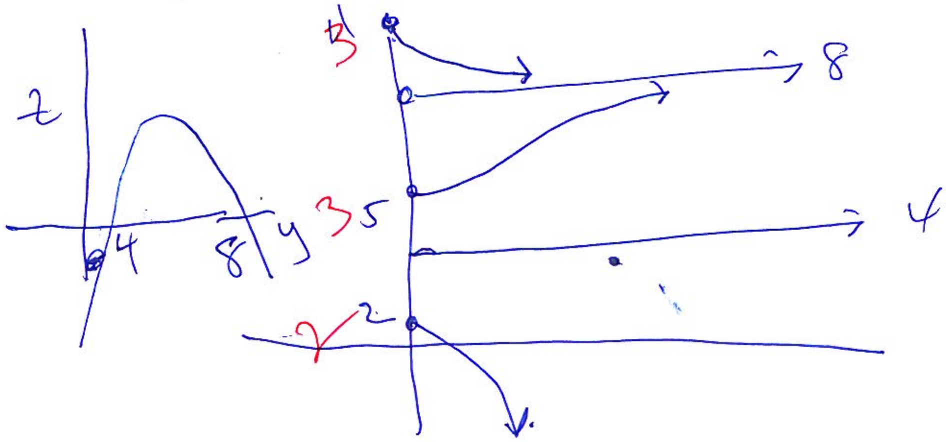
a. Sketch the equilibrium solutions.

8




8

b. Sketch the solutions for the initial conditions y(0) = 2, y(0) = 5, and y(0) = 11.




c. (10 points extra credit) Solve $\frac{dy}{dx}$ =-2 (y -4)( y-8) , y(0) = 5.

10

4 $\int \frac{1}{(y-4)(y-8)}\, dy = \int -2\, dt$

$\frac{1}{(y-4)(y-8)} = \frac{A}{y-4} + \frac{B}{y-8} = -\frac{1}{4}\frac{1}{y-4} + \frac{1}{4}\frac{1}{y-8}$

$1 = A(y-8) + B(y-4)$ 2

$y=8 \quad 1 = B4 \Rightarrow B = \frac{1}{4}$

$y=4 \quad 1 = A(-4) = A = -\frac{1}{4}$

$-\frac{1}{4}\ln|y-4| + \frac{1}{4}\ln|y-8| = -2t + C$

2 $\frac{1}{4}\ln\left|\frac{y-8}{y-4}\right| = -2t + C$

$\frac{1}{4}\ln|3| = C$

2 $\frac{1}{4}\ln\left|\frac{y-8}{y-4}\right| = -2t + \frac{1}{4}\ln 3$

$\ln\left|\frac{y-8}{y-4}\right| = -8t + \ln 3$

$\left|\frac{y-8}{y-4}\right| = 3e^{-8t}$

$\frac{y-8}{y-4} = -3e^{-8t} = w$

$y - 8 = wy - 4w$

$y(1-w) = 8 - 4w$

$y = \frac{8-4w}{1-w} = \boxed{\frac{8+12e^{-8t}}{1+3e^{-8t}}}$