Suppose $f(x) = k(10x+13)/(x^2+5x+6)^2$ for $x \geq 1$ (for $x < 1$, $f(x) = 0$).

a. Find k so that f(x) is a probability density function.

$$\frac{10x+13}{(x+2)^2(x+3)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{x+3} + \frac{D}{(x+3)^2} \Longrightarrow \frac{10x+3}{(x+2)^2(x+3)^2} = \frac{24}{x+2} - \frac{7}{(x+2)^2} - \frac{24}{x+3} - \frac{17}{(x+3)^2}$$

$$10x+13 = A(x+2)(x+3)^2 + B(x+3)^2 + C(x+2)^2(x+3) + D(x+2)^2$$

$x=-2$, $-7 = B(-2+3)^2 = B$

$x=-3$, $-17 = D(-3+2)^2 = D$

coeff of x^3, $0 = A + C$

$x=0$, $13 = 18A + 9B + 12C + 4D$

$13 = 6A + 9(-7) + 4(-17)$

$13 + 63 + 68 = 6A$

$6A = 144$

$A = 24 \Rightarrow C = -24$

$$\frac{1}{k} = \int_1^{+\infty} \frac{24}{x+2} - \frac{7}{(x+2)^2} - \frac{24}{x+3} - \frac{17}{(x+3)^2}\,dx$$

$$\frac{1}{k} = \lim_{b \to +\infty} \left[24\ln|x+2| - 24\ln|x+3| + \frac{7}{x+2} + \frac{17}{x+3} \right]_1^b$$

$$\frac{1}{k} = \lim_{b \to +\infty} \left(24\ln\left|\frac{x+2}{x+3}\right| + \frac{7}{x+2} + \frac{17}{x+3} \right)\Big|_1^b$$

$$= -24\ln\left|\frac{3}{4}\right| - \frac{7}{3} - \frac{17}{4}$$

$$= 24\ln\left|\frac{4}{3}\right| - \frac{79}{12}$$

$$\boxed{k = \frac{1}{24\ln\frac{4}{3} - \frac{79}{12}} = 3.114911527}$$

b. Find the mean of f(x)

$$\mu = k\int_1^{+\infty} \frac{x(10x+13)}{(x+2)^2(x+3)^2}\,dx$$

$$\frac{10x^2+13x}{(x+2)^2(x+3)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x+3} + \frac{D}{(x+3)^2}$$

$$10x^2+13x = A(x+2)(x+3)^2 + B(x+3)^2 + C(x+2)^2(x+3) + D(x+2)^2$$

$x=-2$, $40 - 26 = 14 = B$

$x=-3$ $90 - 39 = 51 = D$

coeff of x^3 $0 = A + C$

$x=0$ $0 = 18A + 9B + 12C + 4D$

$0 = 6A + 9B + 4D$

$6A = -9B - 4D$

$6A = -9(14) - 4(51)$

$6A = -126 - 204 = -330$

$A = -55$

$C = 55$

$$\mu = k\int_1^{+\infty} \frac{-55}{x+2} + \frac{14}{(x+2)^2} + \frac{55}{x+3} + \frac{51}{(x+3)^2}\,dx$$

$$= k\lim_{b \to +\infty} \left[-55\ln|x+2| - \frac{14}{x+2} + 55\ln|x+3| - \frac{51}{x+3} \right]_1^b$$

$$= k\lim_{b \to +\infty} \left(55\ln\left|\frac{x+3}{x+2}\right| - \frac{14}{x+2} - \frac{51}{x+3} \right)\Big|_1^b$$

$$= k\left(-55\ln\frac{4}{3} + \frac{14}{3} + \frac{51}{4} \right)$$

$$= k\left(\frac{209}{12} - 55\ln\frac{4}{3} \right)$$

$= \cancel{15.089\ 10703}$ 4.965644564

c. Consider $g(x) = k(10x^3+13)/(x^2+5x+6)^2$ for $x \geq 1$ (for $x < 1$, $g(x) = 0$).
Explain why g(x) cannot be a probability density function.

For large x, $g(x) \approx \frac{k(10x^3)}{x^4} = \frac{10k}{x}$ and $\int_1^{+\infty} \frac{10k}{x}\,dx$ does not converge

$\cancel{\frac{k(10x^3+13)}{(x^2+5x+6)^2} \geq \frac{k(23x^3)}{(x^2)^2} = \frac{k(23)}{x}}$

$$\frac{k(10x^3+13)}{(x^2+5x+6)^2} \geq \frac{k\,10x^3}{(x^2+5x^2+6x^2)^2} = \frac{10k}{x(12)^2}$$ and $\int_1^{+\infty} \frac{10k}{x(12^2)}\,dx$ diverges

$\therefore$ By comparison $\int_1^{+\infty} k\frac{10x^3+13}{(x^2+5x+6)^2}\,dx$ diverges.