2 Final Exam Name Key
5 Version 2

dit, show all work.

culate the following"

16 a. $\int x^2 \ln(x)\, dx$ $= \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \frac{1}{x}\,dx$

$u = \ln x$

$du = \frac{1}{x}\,dx$ $= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2\,dx$

$dv = x^2\,dx$ $= \boxed{\frac{1}{3}x^3 \ln - \frac{1}{9}x^3 + C}$

$v = \frac{1}{3}x^3$

4 4 4 1 1 1 1

b. $\int (x+1)^2(8-x^2-2x)^{1/2}\,dx$ $= \int (x+1)^2 \left(9-(x+1)^2\right)^{\frac{1}{2}} dx$

16 $w = x+1$, $dw = dx$ $= \int w^2 (9-w^2)^{\frac{1}{2}}\,dw$

$w = 3\sin\theta$ $= \int 3^4 \sin^2\theta \cos^2\theta\, d\theta$

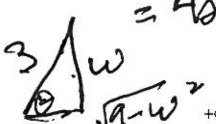
$(9-w^2)^{\frac{1}{2}} = 3\cos\theta\, d\theta$ $= \int \frac{3^4}{2^2}(1-\cos^2 2\theta)\,d\theta$

$dw = 3\cos\theta\, d\theta$ $= \int \frac{3^4}{2^2}\left(1-\left[\frac{1+\cos 4\theta}{2}\right]\right)d\theta$

$\sin 4\theta = 2\sin 2\theta \cos 2\theta$ $= \int \frac{3^4}{2^2}\left(\frac{1}{2} - \frac{\cos 4\theta}{2}\right)d\theta = \int \frac{3^4}{2^3}(1-\cos 4\theta)\,d\theta$

$= 2(2\sin\theta\cos\theta)(1-2\sin^2\theta)$

$= 4\sin\theta\cos\theta(1-2\sin^2\theta)$ $= \frac{3^4}{2^3}\left[\theta - \frac{\sin 4\theta}{4}\right] + C$




$= \frac{3^4}{2^3}\left[\arcsin \frac{w}{3} - \frac{w}{3}\frac{\sqrt{9-w^2}}{3}\left(1-\frac{2w^2}{9}\right)\right] + C$

II. Tell whether $\int_1^{+\infty} \frac{x^5+\cos(e^x)}{x^6+2}\,dx$ converges or diverges, and why.

8 $\frac{x^5+\cos(e^x)}{x^6+2} \geq \frac{x^5}{3x^6} = \frac{1}{3}\frac{1}{x}$ 4

$\int_1^{+\infty} \frac{1}{3}\frac{1}{x}\,dx$ diverges $\Rightarrow \int_1^{+\infty} \frac{x^5+\cos(e^x)}{x^6+2}\,dx$ diverges

$(p=1)$

2 2

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III. Use the Simpson's rule with n = 6 to estimate $\int_4^7 \frac{x}{1+x^6}\,dx$.

4 —— 7

$\Delta x = \frac{7-4}{6} = \frac{3}{6} = \frac{1}{2}$

$\frac{\frac{1}{2}}{3}\left[f(4) + 4f(4.5) + 2f(5) + 4f(5) + 2f(6) + 4f(5.5) + f(6)\right]$

1 1 1 1 7

10

IV. Find the length of the graph of the curve $y = 3 + 8x^{1.5}$, $0 \le x \le 5$.

12 $\int_0^5 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^5 \sqrt{1+\left(12x^{1/2}\right)^2}\,dx = \int_0^5 \sqrt{1+12^2x}\,dx$ 2

4 $= \frac{2}{3}\,\frac{(1+12^2x)^{3/2}}{12^2}\Big|_0^5$ 2

$= \frac{2}{3(12^2)}\left[(1+12^2\cdot 5)^{3/2} - 1\right]$ 2 $\approx 89.$ ~~25.18...~~ 802.449315

V. Find the centroid of the region bounded in the first quadrant by the curves y = x and $32 = x^2 + y^2$. Set up the integrals; you do not have to solve them. (Hint use the region where y = x is the bottom boundary curve.)

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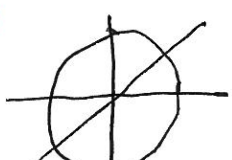

$32 = x^2 + y^2$

$32 = 2y^2$

$16 = x^2$ ✓

4

2 $\text{Mass} = \int_0^4 \left(\sqrt{32-x^2} - x\right)dy$

2 $M_y = \int_0^4 x\left(\sqrt{32-x^2} - x\right)dy$

2 $M_x = \int_0^4 \left(\frac{\sqrt{32-x^2} + x}{2}\right)\left(\sqrt{32-x^2} - x\right)dy$

$\bar{x} = \frac{M_y}{M}$ ✓

$\bar{y} = \frac{M_x}{y}$ ✓

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VI. Find k so that $f(x) = \dfrac{k}{x^2+16x}$ if $x \geq 12$ and $f(x) = 0$ if $x < 12$, is a probability density function.

18 2 $1 = \int_{12}^{+\infty} \frac{k}{x^2+16x}\,dx = \lim_{B\to+\infty} k\frac{1}{16}\ln\left[\frac{x}{x+16}\right]\Big|_{12}^{B}$

$\frac{1}{x(x+16)} = \frac{1}{16}\left[\frac{1}{x} - \frac{1}{x+16}\right]$ 3

$= -k\frac{1}{16}\ln\frac{12}{28}$ 2

$k = \frac{-16}{\ln(\frac{12}{28})}$ 1

$= \frac{16}{\ln(\frac{28}{12})} = 18.88356$

VII. Solve completely:

(a) $\dfrac{dy}{dx} = \dfrac{e^{-y}}{1+4x^2}$ , $y(0) = 1$.

2

$e^y\,dy = \frac{1}{1+4x^2}\,dx$ $u = 2x$ $du = 2dx$

$\int e^y\,dy = \int \frac{1}{1+4x^2}dx = \frac{1}{2}\int\frac{1}{1+u^2}du = \frac{1}{2}\arctan u + C$ 2

2 $e^y = \frac{1}{2}\arctan(2x) + C$ $C = e$

$y = \ln\left(\frac{1}{2}\arctan(2x) + C\right)$ 2 2

(b) $\dfrac{dy}{dx} + 5y = 8e^{-3x}$

10 $I(x) = e^{\int 5\,dx} = e^{5x}$ 2

$\frac{d}{dx}(e^{5x}y) = 8e^{-3x}e^{5x} = 5e^{2x}$ 4

$e^{5x}y = \frac{5}{2}e^{2x} + C$ 2

$y = \frac{5}{2}e^{-3x} + Ce^{-5x}$ 2

(c) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} - 45y = 0.$

10 $r^2 - 4r - 45 = 0$ 3

$(r-9)(r+5) = 0$ 3

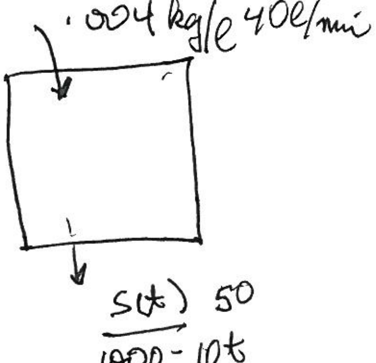
$y(x) = C_1e^{9x} + C_2e^{-5x}$ 4

40

VIII. Use Euler's Method and a stepsize of h = 0.1 to estimate y(.2) where $\frac{dy}{dx} = x(x-5y)^2$, y(0) = 2.

| x | y | $f(x,y)\Delta x$ |
|---|---|---|
| 0 | 2 | $0(0-10)^2(.1) = 0$ |
| .1 | 2 | $.1(.1-10)^2(.1) = .01(-9.9)^2 = .9801$ |
| .2 | 2.9801 | |

IX. A 1000 liter tank is initially filled with brine that contains dissolved salt. A salt solution of .004 kg/l enters the tank at a rate of 40 l/minute; the tank is continuously mixed and a solution drains from the tank at a rate of 50 l/minute. At the start there is exactly 1 kg of salt in the tank. How much salt will be in the tank 20 minutes from the beginning?




$S(0) = 1$

$\frac{dS}{dt}$ = rate in − rate out

$= .16 \text{ kg/min} - \frac{5}{100-t} S(t)$

$\frac{dS}{dt} + \frac{5}{100-t} S(t) = .16$

$I(t) = e^{\int \frac{5}{100-t} dt} = e^{-5\ln(100-t)} = (100-t)^{-5}$

$\frac{d}{dt}\left((100-t)^{-5} S(t)\right) = .16(100-t)^{-5}$

$(100-t)^{-5} S(t) = .16 \frac{(100-t)^{-4}}{4} + C$

$S(t) = \frac{.6}{4}(100-t) + C(100-t)^5$

$1 = S(0) = .15(100) + C(100)^5$

$-14 = C(100)^5$

$C = \frac{-14}{100^5}$

$S(t) = .15(100-t) - \frac{14}{100^5}(100-t)^5$

$S(20) = .15(80) - \frac{14}{100^5}(80)^5 = 12 - 14(.8)^5 = 7.41248$

X. Find the foci and vertices and sketch the graph of $9x^2 - 36x - 16y^2 - 32y = 124$.

$9(x^2 - 4x) - 16(y^2 + 2y) = 124$

$9(x^2 - 4x + 4) - 16(y^2 + 2y + 1) = 124 + 36 - 16$

12 $9(x-2)^2 - 16(y+1)^2 = 12^2$

4 $\dfrac{(x-2)^2}{4^2} - \dfrac{(y+1)^2}{3^2} = 1$

center $= (2, -1)$

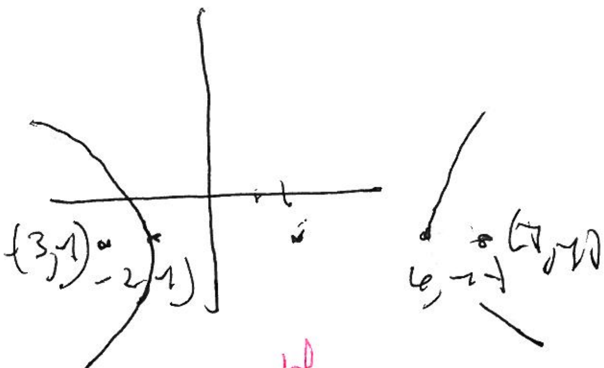
2 vertices $= (2 \pm 4, -1) = \begin{cases} (-2, -1) \\ (6, -1) \end{cases}$

$c^2 = 4^2 + 3^2 = 5^2$

2 foci $= (2 \pm 5, -1) = \begin{cases} (-3, -1) \\ (7, -1) \end{cases}$

asymptotes $(y+1)^2 = \frac{3^2}{4^2}(x-2)^2$

$y + 1 = \pm \frac{3}{4}(x-2)$

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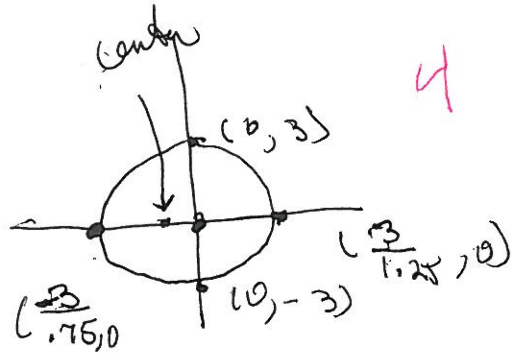
XI. Convert $r = 3/(1+.25\cos(\theta))$ into rectangular coordinates and sketch the graph. Find the slope of the tangent line at $\theta = \frac{\pi}{2}$.

16

5 $r = \dfrac{3}{1 + \frac{1}{4}\cos\theta}$

$r = 3 - \frac{1}{4} r\cos\theta$

6 $x^2 + y^2 = (3 - \frac{1}{4}x)^2$

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at (0,3) $2x + 2y\frac{dy}{dx} = 2(3 - \frac{1}{4}x)(-\frac{1}{4})$

$2(3)\frac{dy}{dx} = -\frac{1}{2}(3)$

2 $\frac{dy}{dx} = -\frac{1}{4}$

XII. For $y = t^3 + 3t^2 - 9t$ and $x = t^4 - 8t^2$, $-4 < t < 3$

(a) Find the points where the parametric system has a vertical tangent line.

$$\frac{dx}{dt} = 4t^3 - 16t = 4t(t^2-4) = 4t(t-2)(t+2) = 0$$ 3

| t | x | y |
|---|---|---|
| -2 | -16 | 22 |
| 0 | 0 | 0 |
| 2 | -16 | 2 |

3

6

(b) Find the points where there are horizontal tangent lines.

$$\frac{dy}{dt} = 3t^2 + 6t - 9 = 3(t^2 + 2t - 3) = 3(t+3)(t-1)$$ 3

6

| t | x | y |
|---|---|---|
| -3 | 9 | 27 |
| 1 | -7 | -5 |

3

(c) Find where x is increasing.

x dec, x inc, x dec, x inc
-2, 0, 2

3

x is inc on $(-2, 0) \cup (2, +\infty)$

3

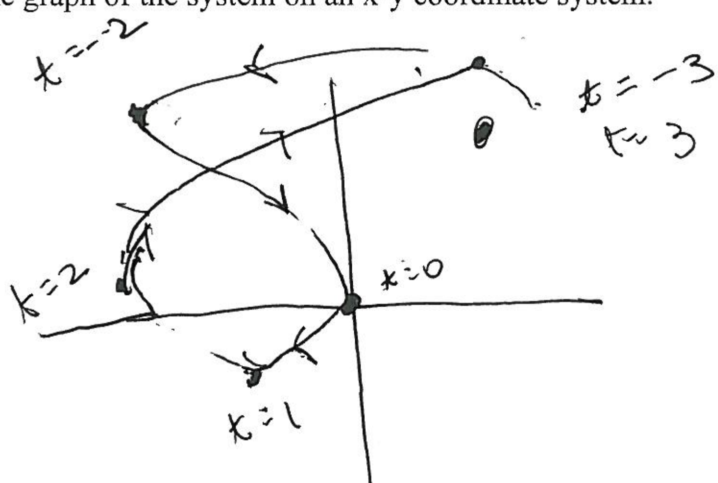
(d) Find where y is increasing.

y is inc, y is dec, y is inc
-3, 1

y is inc on $(-\infty, -3) \cup (1, +\infty)$ 3

3

(e) Sketch the graph of the system on an x-y coordinate system.

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XIII. Tell why each series is conditionally convergent, absolutely convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^4+5}$ absolutely convergent

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(b) $\sum_{n=1}^{\infty} (-1)^n 9^{-n} \cos(e^{-n})$ absolutely convergent

3/3

(c) $\sum_{n=1}^{\infty} \frac{\sin(e^{-n})}{e^{-n}}$ divergent $\lim_{n \to +\infty} \frac{\sin e^{-n}}{e^{-n}} = 1 \neq 0$

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XIV. Find the radius and interval of convergence for $f(x) = \sum_{n=1}^{\infty} n(4x+12)^n 8^{2n}$ .

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n+1\,(4x+12)^{n+1}\,8^{2n+2}}{n\,(4x+12)^n\,8^{2n}}\right| = \frac{n+1}{n}\,|4x+12|\,8^2 \rightarrow |4x+12|\,8^2 < 1$$

3 3

14

$|x+3|\,64 < 1$

$-\frac{1}{16} < x+3 < \frac{1}{16}$

$-3-\frac{1}{16} < x < -3+\frac{1}{16}$

$R = \frac{1}{16}$

at $x = -3+\frac{1}{16}$ $\quad \sum n\,\frac{1}{4^n}\,8^{2n} = \sum n\,16^n$

doesn't converge

at $x = -3-\frac{1}{16}$ $\quad \sum n\left(-\frac{1}{4^n}\right)8^{2n} = \sum n(-1)^n 16^n$

does not converge

$\left(-3-\frac{1}{16}, -3+\frac{1}{16}\right)$ 2

XV. Use a power series to estimate $\int_0^{0.1} \frac{\sin(x^7)-x^7}{4x^{10}}\,dx$ with an error less than $10^{-25}$.

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$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\sin x^7 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x^7)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{14n+7}$$

$$\sin x^7 - x^7 = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{14n+7}$$

$$\frac{\sin x^7 - x^7}{4x^{10}} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{14n-3}$$

$$\int_0^{0.1} \frac{\sin x^7 - x^7}{4x^{10}} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{(-1)^n\, x^{14n-2}}{(2n+1)!\,(14n-2)}\Bigg|_0^{.1}$$ 2

$$= \frac{1}{4}\sum_{n=1}^{\infty} \frac{(-1)^n (.1)^{14n-2}}{(2n+1)!(14n-2)}$$ 2 $n=2$ less than $10^{-25}$

$$= \frac{1}{4}\,\frac{(-1)^1 (.1)^{12}}{3!\,(12)}$$ 2

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