MAT 162 Final Exam Name Key

Spring 2015 Version 1

248 points

For full credit, show all work.

I. Calculate the following”

16 a. $\int x^2 \sin(3x)\, dx = -\frac{x^2}{3}\cos 3x - \int -\frac{\cos(3x)}{3}\, 2x\,dx = -\frac{x^2}{3}\cos 3x + \frac{2}{3}\int x\cos(3x)\,dx$ (2) (2)

4 $\left\{ \begin{array}{l} u = x^2 \\ du = 2x\,dx \\ dv = \sin(3x)\,dx \\ v = -\frac{\cos(3x)}{3} \end{array}\right.$

$= -\frac{x^2}{3}\cos(3x) + \frac{2}{3}\left[x\frac{\sin(3x)}{3} - \int \frac{\sin(3x)}{3}\,dx\right]$ (2) (2)

$= -\frac{x^2}{3}\cos(3x) + \frac{2}{9}x\sin(3x) + \frac{2}{9}\frac{\cos(3x)}{3} + C$

$u = x \quad dv = \cos(3x)$

$du = dx \quad v = \frac{\sin(3x)}{3}$

$= -\frac{x^2}{3}\cos(3x) + \frac{2}{9}x\sin(3x) + \frac{2}{27}\cos(3x) + C$ (1) (1) (1) (1)

16 b. $\int x^2(9-x^2)^{1/2}\,dx = \int 3^2\sin^2\theta\; 3\cos\theta\; 3\cos\theta\, d\theta$ (3)

3 $\left\{ \begin{array}{l} x = 3\sin\theta \\ (9-x^2)^{\frac{1}{2}} = 3\cos\theta \\ dx = 3\cos\theta\, d\theta \end{array}\right.$

$= \int 3^4\sin^2\theta\cos^2\theta\, d\theta$ (2) or $= \int 3^4\left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cos 2\theta}{2}\right)d\theta$ (2)

$= \int 3^4(\sin\theta\cos\theta)^2\, d\theta$ $\qquad = \int \frac{3^4}{2^2}(1-\cos^2 2\theta)\,d\theta$ (1)

$= \int 3^4\left(\frac{\sin 2\theta}{2}\right)^2 d\theta$ $\qquad = \int \frac{3^4}{2^2}\left(1-\left(\frac{1+\cos 4\theta}{2}\right)\right)d\theta$ (1)

$= \int \frac{3^4}{2^2}\sin^2 2\theta\, d\theta$ $\qquad = \int \frac{3^4}{2^2}\left(\frac{1}{2} - \frac{\cos 4\theta}{2}\right)d\theta$ (1)

$= \int \frac{3^4}{2^2}\,\frac{1-\cos 4\theta}{2}\, d\theta$ $\qquad = \int \frac{3^4}{2^3}(1-\cos 4\theta)\,d\theta$ (1)

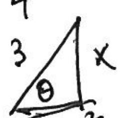
$\sin 4\theta = 2\sin 2\theta\cos 2\theta$

$= 2(2\sin\theta\cos\theta)(1-2\sin^2\theta)$

$= 4\sin\theta\cos\theta(1-2\sin^2\theta)$

$\frac{\sin 4\theta}{4} = \sin\theta\cos\theta(1-2\sin^2\theta)$

$= \frac{3^4}{2^3}\left[\theta - \frac{\sin 4\theta}{4}\right] + C$ (1)



$= \frac{3^4}{2^3}\left[\arcsin\frac{x}{3} - \frac{x}{3}\frac{\sqrt{9-x^2}}{3}\left(1-2\frac{x^2}{9}\right)\right] + C$ (1) (1)

II. Tell whether $\int_1^{+\infty} \frac{x^2+\cos(e^x)}{x^3+2}\,dx$ converges or diverges, and why.

8

$\frac{x^2+\cos(e^x)}{x^3+2} \geq \frac{x^2}{3x^3} = \frac{1}{3}\frac{1}{x}$ (4)

$\int_1^{+\infty}\frac{1}{3}\frac{1}{x}\,dx$ diverges $\Rightarrow \int_1^{+\infty}\frac{x^2+\cos(e^x)}{x^3+2}\,dx$ diverges (2)

$p = 1$ (2)

40

III. Use the Trapezoidal rule with n = 6 to estimate $\int_4^7 \frac{x}{1+x^6}\,dx$.

4 7

$\Delta x = \frac{7-4}{6} = \frac{3}{6} = \frac{1}{2}$ 1

$\frac{\frac{1}{2}}{2}\Big[f(4) + 2f(4.5) + 2f(5) + 2f(5.5) + 2f(6) + 2f(6.5) + f(7)\Big]$

1 ✓

7

10

IV. Find the length of the graph of the curve $y = 17 + 24x^{1.5}$, $0 \le x \le 5$.

12

$\int_0^5 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^5 \sqrt{1+(36x^{\frac{1}{2}})^2}\,dx = \int_0^5 \sqrt{1+36^2 x}\,dx$ 2 2

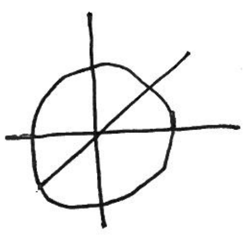
$= \frac{2}{3}\frac{(1+36^2 x)^{3/2}}{36^2}\Big|_0^5$ 2

$= \frac{2}{3(36^2)}\left[(1+36^2 5)^{3/2} - 1\right]$ 2

$= 268.3847583$

V. Find the centroid of the region bounded in the first quadrant by the curves y = x and 18 = x² + y². Set up the integrals; you do not have to solve them. (Hint use the region where y = x is the bottom boundary curve.)

12



$18 = x^2 + x^2$

$18 = 2x^2$

$x = 3$

2

2 $Mass = \int_0^3 (\sqrt{18-x^2} - x)\,dx = 7.068583471$

2 $M_y = \int_0^3 x(\sqrt{18-x^2} - x)\,dx = 7.455844123$

2 $M_x = \int_0^3 \left(\frac{\sqrt{18-x^2}+x}{2}\right)\left(\sqrt{18-x^2} - x\right)dx$ $= 18$

2 $\bar{x} = \frac{M_y}{M} = 1.0548$

2 $\bar{y} = \frac{M_x}{M} = 2.54648$

34

2 2

VI. Find k so that $f(x) = \frac{k}{x^2+12x}$ if $x \geq 10$ and $f(x) = 0$ if $x < 10$, is a probability density function.

(10) 1 2 $\int_{10}^{+\infty} \frac{k}{x(x+12)}\,dx = \lim_{b\to+\infty} k\frac{1}{12}\ln\left(\frac{x}{x+12}\right)\Big|_{10}^{b} = k\frac{1}{12}\ln\left(\frac{22}{10}\right) = k\frac{1}{12}\ln 2.2$

$\frac{1}{x(x+12)} = \frac{A}{x} + \frac{B}{x+12} = \frac{1}{12}\frac{1}{x} - \frac{1}{12}\frac{1}{x+12}$

$1 = A(x+12) + Bx$

$x=-12 \quad 1 = B(-12) \Rightarrow B = -\frac{1}{12}$

$1 = A(12) \Rightarrow A = \frac{1}{12}$ 3

$k = \frac{12}{\ln 2.2}$ 1

$= 15.21958284$

VII. Solve completely:

(a) $\frac{dy}{dx} = \frac{1+e^{-y}}{1+4x^2}$, $y(0) = 1$.

(10)

$u = 2x$
$du = 2dx$

2 $\frac{1}{1+e^{-y}}\,dy = \frac{1}{1+4x^2}\,dx$

$\int \frac{e^y}{e^y+1}\,dy = \int \frac{1}{1+(2x)^2}\,dx = \int \frac{1}{1+u^2}\frac{du}{2} = \frac{1}{2}\arctan u = \frac{1}{2}\arctan 2x + C$

2 $\ln(1+e^y) = \frac{1}{2}\arctan 2x + C$ 2 $\quad \ln(1+e) = \frac{1}{2}(0) + C$

1 $1+e^y = e^{\frac{1}{2}\arctan 2x + C} = e^{\frac{1}{2}\arctan 2x + \ln(1+e)} = (1+e)e^{\frac{1}{2}\arctan(2x)}$

~~$y = (1+e)e^{\frac{1}{2}\arctan 2x}$~~

1 $y = \ln\left((1+e)e^{\frac{1}{2}\arctan(2x)} - 1\right)$

(b) $\frac{dy}{dx} + 8y = 5e^{-3x}$

(10)

$I(x) = e^{\int 8\,dx} = e^{8x}$ 2

$\frac{d}{dx}(e^{8x}y) = 5e^{8x}e^{-3x} = 5e^{5x}$ 4

$e^{8x}y = e^{5x} + C$ 2

$y = e^{-3x} + Ce^{-8x}$ 2

(c) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 32y = 0$.

(10)

$r^2 + 4r - 32 = 0$ 3

$(r+8)(r-4) = 0$ 3

$y(x) = c_1e^{-8x} + c_2e^{4x}$ 4

40

VIII. Use Euler's Method and a stepsize of h = 0.1 to estimate y(.2) where $\frac{dy}{dx} = x(x+5y)^2$, y(0) = 2.

x	y	f(x,y)(.1)
0	2	$0(0+5(2))^2(.1)=0$ 2
.1	2	$.1(.1+10)^2(.1) = .01(10.1)^2 = 1.0201$ 2
.2	3.0201	

2 2 2 10

IX. A 1000 liter tank is initially filled with brine that contains dissolved salt. A salt solution of .004 kg/l enters the tank at a rate of 50 l/minute; the tank is continuously mixed and a solution drains from the tank at a rate of 60 l/minute. ~~In 10 minutes from~~ At the start there is exactly ~~1~~ 9 kg of salt in the tank. How much salt will be in the tank 30 minutes from the beginning?

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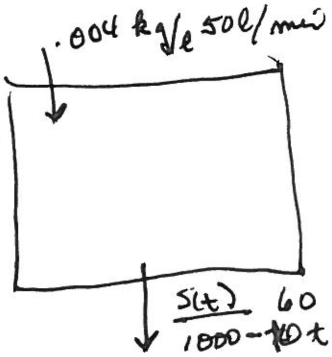


$S(\cancel{10}\,0) = 1$ kg 2

$\frac{ds}{dt} = .004(50) - \frac{60}{1000-10t}S(t)$ .2

$\frac{ds}{dt} + \frac{6}{100-t}S(t) = .2$ 2

$I(t) = e^{\int \frac{6}{100-t}dt} = e^{-6\ln(100-t)} = (100-t)^{-6}$ .2

2 $\frac{d}{dt}\left((100-t)^{-6}S(t)\right) = .2(100-t)^{-6}$

2 $(100-t)^{-6}S(t) = \frac{.2(100-t)^{-5}}{5} + c$

2 $S(t) = \frac{1}{25}(100-t) + c(100-t)^6$

$1 = S(\cancel{10}) = \frac{1}{25}(\cancel{90}) + c(\cancel{90}\,100)^6 \Rightarrow -\frac{3}{100^6} = c$

2 $-2.6 = c(90)^6$

$c = \frac{-2.6}{(90)^6}$

$S(t) = \frac{1}{25}(100-t) - \frac{2.6}{90^6}(100-t)^6$

$S(30) = \frac{1}{25}(100-30) - \frac{2.6}{90^6}(100-30)^6$

2 $= \ldots\ 2.2241889\ \text{kg}$

$S(t) = \frac{1}{25}(100-t) - \frac{3}{100^6}(100-t)^6$

$S(30) = \ldots\ 3(.7)^6$

2.44705^3

−30

X. Find the foci and vertices and sketch the graph of $x^2 - 6x - 4y^2 + 16y = 32$.

$x^2 - 6x + 9 - 4(y^2 - 4y + 4) = 32 + 9 - 16 = 41 - 16 = 25$

13 4 $\frac{(x-3)^2}{5^2} - \frac{(y-2)^2}{\left(\frac{5}{2}\right)^2} = 1$ $\left(\frac{5}{2}\right)^2 \frac{(x-3)^2}{5^2} = (y-2)^2$

$y - 2 = \pm \frac{1}{2}(x-3)$

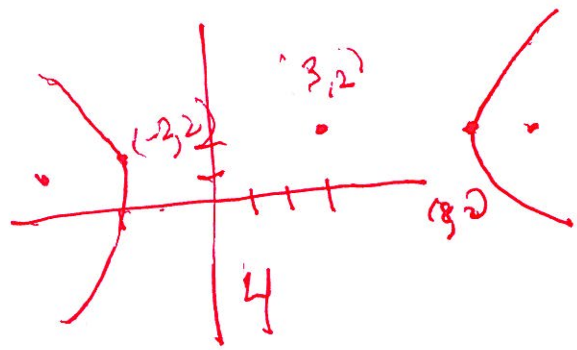
center $= (3, 2)$

vertices $= (3 \pm 5, 2) = \begin{cases} (8, 2) \\ (-2, 2) \end{cases}$

2 $c^2 = a^2 + b^2 = 5^2 + \left(\frac{5}{2}\right)^2 = 5^2\left(\frac{5}{4}\right)$

$c = \frac{5\sqrt{5}}{2}$

2 foci $= \left(3 \pm \frac{5\sqrt{5}}{2}, 2\right) = \begin{cases} \left(3 + \frac{5\sqrt{5}}{2}, 2\right) \\ \left(3 - \frac{5\sqrt{5}}{2}, 2\right) \end{cases}$



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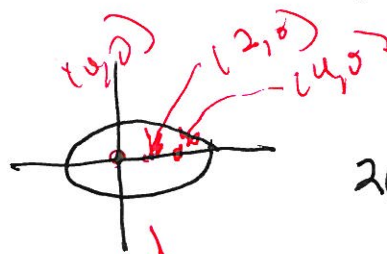
XI. Convert $r = 3/(1 - .5\cos(\theta))$ into rectangular coordinates and sketch the graph. Find the slope of the tangent line at $\theta = \frac{\pi}{2}$.

16



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$r - .5r\cos\theta = 3$

$r = 3 + .5r\cos\theta$

$r^2 = (3 + .5r\cos\theta)^2$

$x^2 + y^2 = (3 + .5x)^2$

$x^2 + y^2 = 9 + 3x + .25x^2$

$.75x^2 - 3x + y^2 = 9$

8

$\frac{3}{4}x^2 - 3x + y^2 = 9$

$\frac{3}{4}(x^2 - 4x) + y^2 = 9$

$\frac{3}{4}(x^2 - 4x + 4) + y^2 = 12$

$\frac{(x-2)^2}{4\left(\frac{4}{3}\right)} + \frac{y^2}{12} = 1$

$\frac{(x-2)^2}{16} + \frac{y^2}{12} = 1$ $\frac{4}{16} + \frac{9}{12}$

$\frac{2(x-2)}{16} + \frac{2y\,dy}{12\,dx} = 0$

$\frac{dy}{dx} = \frac{-2(x-2)}{16}\,\frac{12}{2y}$

4

$= \frac{-2(-2)}{16}\,\frac{12}{2(3)}$

$m = \frac{1}{2}$

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XII. For $x = t^3 + 3t^2 - 9t$ and $y = t^4 - \cancel{4t^2}\ 8t^2$, $-4 < t < 3$

(a) Find the points where the parametric system has a vertical tangent line.

$\frac{dx}{dt} = 3t^2 + 6t - 9 = 3(t^2 + 2t - 3) = 3(t+3)(t-1)$ 3

$t = -3$

$t = 1$

t	x	y
-3	27	9
1	-5	-7

2

6

(b) Find the points where there are horizontal tangent lines.

$\frac{dy}{dt} = 4t^3 - 16t$ 2

$= 4t(t^2 - 4)$

$= 4t(t-2)(t+2)$ 2

t	x	y
-2	22	-16
0	0	0
2	2	-16

2

6

(c) Find where x is increasing.

$\frac{dx}{dt} > 0$ $\quad \frac{dx}{dt} < 0$ $\quad \frac{dx}{dt} > 0$ 3

x: -3, 1

x is inc on $(-\infty, -3) \cup (1, +\infty)$ 3

y is dec on $(-3, 1)$

0

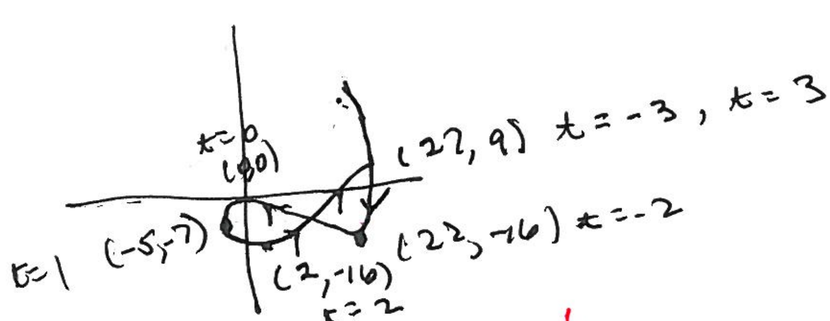
(d) Find where y is increasing.

y dec, y inc, y dec, y inc: -2, 0, 2 3

y is inc on $(-2, 0)$ and $(2, +\infty)$ 3

0

(e) Sketch the graph of the system on an x-y coordinate system.



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XIII. Tell why each series is conditionally convergent, absolutely convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^3 + 5}$ Conditionally convergent 3

reasons 3

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n 9^n \sin\left(e^{-\overset{n}{\cancel{2}}}\right)$ divergent 3

1

reasons 3

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n^4)}{n^3}$ absolutely convergent 3

reasons 3

18

XIV. Find the radius and interval of convergence for $f(x) = \sum_{n=1}^{\infty} n(4x+12)^n 8^{-2n}$

$$\left| \frac{(n+1)(4x+12)^{n+1}}{(8^2)^{n+1}} \cdot \frac{(8^2)^n}{n(4x+12)^n} \right| = \frac{n+1}{n} \frac{(4x+12)}{8^2} \rightarrow \frac{|4x+12|}{8^2} < 1$$ 3 3

14

$|4x+12| < 64$

$-64 < 4x+12 < 64$

$-76 < 4x < 52$

$-19 < x < 13$

4

at 13 $\sum n \frac{(64)^n}{64^n}$ diverges 1

at -19 $\sum n \frac{(-64)^n}{64^n}$ diverges 1

$R = \frac{32}{2} = 16$ 8

$\frac{13+19}{2} = \frac{32}{2} = 16$

$(-19, 13)$ 2

XV. Use a power series to estimate $\int_0^{0.1} \frac{\sin(x^8) - x^8}{4x^3} dx$ with an error less than 10^{-25}.

2 $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

2 $\sin x^8 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^8)^{2n+1}}{(2n+1)!}$

2 $\sin x^8 - x^8 = \sum_{n=1}^{\infty} \frac{(-1)^n (x^8)^{2n+1}}{(2n+1)!}$

2 $\frac{\sin x^8 - x^8}{4x^3} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{(-1)^n x^{16n+5}}{(2n+1)!}$

$\int_0^{0.1} \frac{\sin x^8 - x^8}{4x^3} dx = \frac{1}{4} \sum_{n=1}^{\infty} \frac{(-1)^n x^{16n+6}}{(2n+1)!(16n+6)} \Big|_0^{.1}$ 2

14

$= \frac{1}{4} \sum_{n=1}^{\infty} \frac{(-1)^n (.1)^{16n+6}}{(2n+1)!(16n+6)}$ 2

$n=1$ $\boxed{-\frac{1}{4} \frac{(.1)^{22}}{3!(22)}}$ since $|a_2| < 10^{-25}$ 2

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