MAT 162 Final Exam Name Key
Fall 2014 Version 2

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For full credit, show all work.

I. Calculate the following”

a. $\int x\sin^2(x)\ dx$ $= \int x\left(\frac{1-\cos 2x}{2}\right)dx = \frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos 2x\,dx$ 3 2

(15)

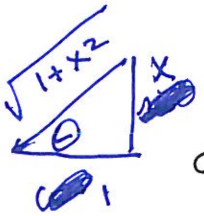
$= \frac{1}{2}\frac{x^2}{2} - \frac{1}{2}\left[x\left(\frac{\sin 2x}{2}\right) - \int \frac{\sin 2x}{2}dx\right]$ 2 2 2 2

$= \boxed{\frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C}$ 1 1 1 1

check: $\frac{1}{2}x - \frac{1}{4}\sin 2x - \frac{1}{4}x\cos 2x(2) + \frac{1}{8}\sin 2x)2 = x\left(\frac{1-\cos 2x}{2}\right)$
$= x\sin^2 2x$

b. $\int x^3(1+x^2)^{1/2}dx$ $= \int \tan^3\theta\ \sec\theta\ \sec^2\theta\,d\theta = \int \tan^2\theta\ \sec^2\theta\ \tan\theta\sec\theta\,d\theta$ 1 1 1

$x = \tan\theta$ 1 $= \int(\sec^2\theta - 1)\sec^2\theta\ \tan\theta\sec\theta\,d\theta$ 1



$w = \sec\theta$ 1 $= \int(w^2-1)w^2\,dw$ 1

$dw = \sec\theta\tan\theta\,d\theta$ 1 $= \frac{1}{5}w^5 - \frac{1}{3}w^3 + C$ 1

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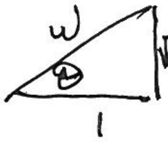

$w = \frac{1}{\cos\theta}$ 1 $= \frac{1}{5}\sec^5\theta - \frac{1}{3}\sec^3\theta + C$ 1

$\frac{1}{w} = \cos\theta$

$= \boxed{\frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + C}$

$\sec^2\theta = 1 + \tan^2\theta = 1 + x^2$ 1

$w = \sec\theta = \sqrt{1+x^2}$ 1

2

II. Tell whether $\int_1^{+\infty}\frac{1+\cos(\sin(5x^3))}{x^3+17}dx$ converges or diverges, and why.

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3

$$\frac{1+\cos(\sin(5x^3))}{x^3+17} \le \frac{2}{x^3}$$

$\int_1^{+\infty}\frac{2}{x^3}dx$ converges $\Rightarrow$ $\int_1^{+\infty}\frac{1+\cos(\sin(5x^3))}{x^3+17}dx$ converges

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3 $p = 3$ ~~series~~ integral

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III. Use the Simpson's rule with n = 4 to estimate $\int_4^7 \frac{x}{1+x^6}\,dx$.

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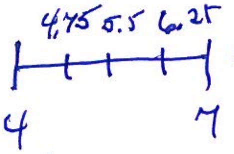

1

$f(x) = \frac{x}{1+x^6}$

$2\ \frac{\Delta x}{3}\left(f(4) + 4f(4.75) + 2f(5.5] + 4f(6.25]\right)$ 1 1 1 1

$\Delta x = \frac{7-4}{4} = \frac{3}{4}$ 2

$\frac{1}{4}\left(f(4) + 4f(4.75) + 2f(5.5) + 4f(6.25) + f(7)\right) + f(7)$ 1

$= 8.431807886 \times 10^{-4}$

0

(calculator $\int_4^7 \frac{x}{1+x^6}dx = 8.723442192 \times 10^{-4}$)

IV. Find the length of the graph of the curve $y = 9 + 8x^{1.5}$, $0 \le x \le 3$.

$f(x) = 9 + 8x^{1.5}$

10

$f'(x) = 12x^{.5}$

$1\ \int_0^3 \sqrt{1+(12x^{.5})^2}\,dx = \int_0^3 \sqrt{1+144x}\,dx = \frac{2}{3}\frac{(1+144x)^{3/2}}{144}\Big|_0^3$ 1 1 1 1 1 1 ~~2~~ 1

$= \frac{2}{3(144)}(433^{3/2} - 1)$ 1

in the first and second quadrant

$= 41.70901082$

V. Find the centroid of the region bounded by the curves $y = x^2$ and $20 = x^2 + y^2$. Set up the integrals; you do not have to solve them.

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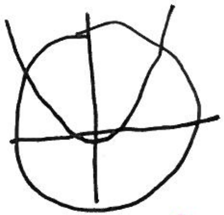

$20 = y + y^2$

$0 = y^2 + y - 20$

$= (y+5)(y-4)$

$y = 4$

$x = \pm 2$

②

0

1 ⓪ top $y = x^3$

1 ⓪ bottom $20 = x^3 + y^2$

$\text{Mass} = \int_{-2}^{2} \rho\left(\sqrt{20-x^2} - x^2\right)dx$ 2

$M_y = \int_{-2}^{2} \rho\, x\left(\sqrt{20-x^2} - x^2\right)dx$ 2

$M_x = \int_{-2}^{2} \frac{\rho}{2}\left((20-x^2) - x^4\right)dx$ 2

$\bar{x} = \frac{~~\text{Mass}~~\ M_y}{~~\text{Mass}~~\ \text{Mass}}$ 2

$\bar{y} = \frac{~~\text{Mass}~~\ M_x}{~~M_x~~\ \text{Mass}}$ 2

/35

VI. Find k so that $f(x) = \frac{k}{x^2+10x}$ if x≥4 and f(x) = 0 if x <4, is a probability density function. 2

10 $1 = \int_4^{+\infty} \frac{k}{x^2+10x}\,dx = k\int_4^{+\infty} \frac{1}{x(x+10)}\,dx = k\int_4^{+\infty} \frac{1}{10}\frac{1}{x} - \frac{1}{10}\frac{1}{x+10}\,dx$

2

$\frac{1}{x(x+10)} = \frac{A}{x} + \frac{B}{x+10}$

$1 = A(x+10) + Bx$

$1 = A(10) \Rightarrow A = \frac{1}{10}$

$1 = B(-10) \Rightarrow B = -\frac{1}{10}$

$= k \lim_{b\to+\infty} \frac{1}{10}\ln x \Big|_4^b - \frac{1}{10}\ln|x+10|\Big|_4^b$ 2

$= k \lim_{b\to+\infty} \frac{1}{10}\ln\left|\frac{x}{x+10}\right|\Big|_4^b = 0 - \frac{1}{10}\ln\frac{4}{14}$

$= k$ ⇒ $k = \frac{10}{\ln\frac{14}{4}}$ 2

$\frac{1}{10}\ln\frac{14}{4}$

2

VII. Solve completely:

10 (a) $\frac{dy}{dx} = \frac{1+y}{1+8x^2}$, y(0) = 1.

$\frac{1}{1+y}\,dy = \frac{1}{1+8x^2}\,dx$ 2

$\ln|1+y| = \frac{1}{4}\arctan(4x) + c$ 2

$\ln 2 = \frac{1}{4}(0) + c \Rightarrow c = \ln 2$ 1

$\ln|1+y| = \frac{1}{4}\arctan(4x) + \ln 2$ 1

2 $|1+y| = 2e^{\frac{1}{4}\arctan(4x)}$

$y = -1 + 2e^{\frac{1}{4}\arctan(4x)}$ 2

(b) $\frac{dy}{dx} + 6y = 5e^{3x}$

10 $I(x) = e^{6x}$ 3

$\frac{d}{dx}(ye^{6x}) = 5e^{9x}$ 2

$ye^{6x} = \frac{5}{9}e^{9x} + C$ 3

$y = \frac{5}{9}e^{3x} + Ce^{-6x}$ 2

(c) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 12y = 0.$ 3

10 $r^2 - 4r - 12 = 0$ 3

$(r-6)(r+2) = 0$

$r = 6,\ r = -2$ 2

$y = c_1e^{6x} + c_2e^{-2x}$ 2

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VIII. Use Euler's Method and a stepsize of h = 0.1 to estimate y(.2) where $\frac{dy}{dx} = x(1+3y)^2$, y(0) = 2.

x	y	$\Delta y = [x(1+3y)^2](.1)$	
1 0	2 1	$0(.1) = 0$	2
1 .1	2 1	$[.1(7^2)](.1) = .49$	2
1 .2	2.49 1		

(10)

$$\cancel{1 = S(10) = \frac{.2}{36}(510)}^{35} \quad \cancel{+ 50\left(\frac{500}{510}\right)^{35} - \frac{100}{36}\left(\frac{500}{510}\right)}$$

$$\cancel{\frac{1 - \frac{.2(510)}{36} + \frac{100}{36}\left(\frac{500}{510}\right)^{35}}{\left(\frac{500}{510}\right)^{35}}} = S_0$$

IX. A 1000 liter tank is initially filled with brine that contains dissolved salt. A salt solution of .004 kg/l enters the tank at a rate of 50 l/minute; the tank is continuously mixed and a solution drains from the tank at a rate of 70 l/minute. In 10 minutes there is exactly 1 kg of salt in the tank. How much salt was in the tank in the beginning?

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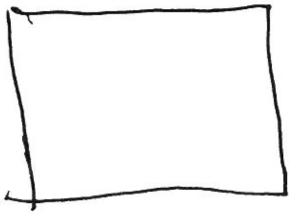

$S(t)$ = amount of salt in tank

$S(0) = S_0$ unknown

2 $\frac{dS}{dt}$ = rate in − rate out

~~$1 = \frac{.2}{36}(500) + S_0 \left(\frac{.2(500)}{36}\right)$~~

2 $= 50(.004)$ kg/min $- \frac{S}{1000 - 20t} 70$

$= 5(.04) - \frac{35}{500 - t} S$

2 $= .2 - \frac{35}{500 - t} S$

2 $\frac{dS}{dt} + \frac{35}{500 - t} S = .2$

2 $I(t) = e^{-35 \ln(500 - t)} = (500 - t)^{-35}$

2 $\frac{d}{dt}\left(S(t)(500 - t)^{-35}\right) = .2(500 - t)^{-35}$

2 $S(t)(500 - t)^{-35} = \frac{.2}{34}(500 - t)^{-34} + C$

2 $S(t) = \frac{.2}{34}(500 - t) + C(500 - t)^{35}$ .08

$1 = S(10) = .08(40) + C(40)^{3.5} \Rightarrow C = \frac{-2.2}{40^{3.5}}$

$S(0) = .08(50) - 2.2\left(\frac{5}{4}\right)^{3.5} \rightarrow 4 - 2.2(1.25)^{3.5} = -.8840522.95$

~~$S_0(500)^{35} = \frac{.2}{36}(500) + C$~~

~~$S_0(500)^{35} - \frac{.2}{36}(500) = C$~~

~~$S(t) = \frac{.2}{36}(500+t) + S_0\left(\frac{500}{500+t}\right)^{35} - \frac{.2(500)}{36}\left(\frac{500}{500+t}\right)^{35}$~~

2 $1 = S(10) = \frac{.2}{34}(490) + C(490)^{35}$

C $\frac{1 - \frac{490}{170}}{(490)^{35}}$

$S(t) = \frac{1}{170}(500 - t) + C(500 - t)^{35}$

2 $S_0 = S(0) = \frac{1}{170}(500) + \left(1 - \frac{490}{170}\right)\left(\frac{500}{490}\right)^{35}$

$= -.8764058232$

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X. Find the foci and vertices and sketch the graph of $x^2 -6x + 4y^2 + 16y = -21$.

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$x^2-6x+9+4(y^2+4y+4) = -21+9+36 = \cancel{28}\ 4$

1 1

$\frac{(x-3)^2}{\cancel{28}\ 4} + \frac{(y+2)^2}{\cancel{\frac{28}{4}}\ 1} = 1$ 3

center $(3,-2)$ ellipse

$c^2 = \cancel{28}\,4 - \cancel{\frac{28}{4}}\,1 = \cancel{28(\frac{3}{4})}\,3$

$c = \cancel{\ }\sqrt{3}$

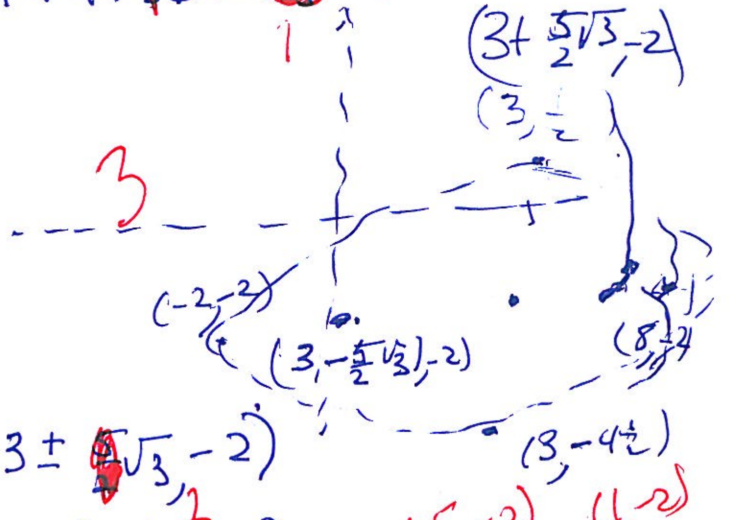
$a = \cancel{5}\ 2$

$b = \cancel{\frac{5}{2}}\ 1$

2 foci $= (3 \pm \cancel{\frac{5}{2}}\sqrt{3}, -2)$

2 major vertex $= (3 \pm \cancel{5}\,2, -2 = (5,-2)\ (1,-2)$

2 minor vertex $= (3, -2 \pm \cancel{\frac{5}{2}}\,1)$ $(3,-1)\ (3,-3)$



XI. Convert r = 2/(1+.5sin(θ)) into rectangular coordinates and sketch the graph. Find the slope of the tangent line at $\theta = \frac{\pi}{2}$.

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$r + .5r\sin\theta = 2$

$r = 2 - .5r\sin\theta$

$r^2 = (2-.5r\sin\theta)^2$

$x^2+y^2 = (\cancel{\ }\,2-.5y)^2$

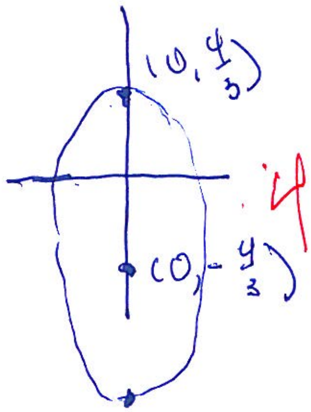
$x^2+y^2 = 4 - \cancel{\ }2y + \frac{1}{4}y^2$

$x^2 + \frac{3}{4}y^2 + 2y = 4$

$x^2 + \frac{3}{4}(y^2 + \frac{8}{3}y + (\frac{4}{3})^2) = 4 + \frac{4}{3} = \frac{16}{3}$

$\frac{x^2}{\frac{16}{3}} + \frac{(y+\frac{4}{3})^2}{\frac{16}{3}\ \frac{4}{3}} = 1$

3 $\frac{x^2}{\frac{16}{3}} + \frac{(y+\frac{4}{3})^2}{(\frac{8}{3})^2} = 1$



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$2x + \frac{3}{2}y\frac{dy}{dx} + 2\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x}{2+\frac{3}{2}y}$

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$= \frac{-2(0)}{2+\frac{3}{2}(\frac{4}{3})}$

$= \boxed{0}$

or

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$

at $\theta = \frac{\pi}{2}$

$= \frac{\frac{dr}{d\theta}}{-r} = \frac{0}{-\frac{4}{3}} = 0$

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XII. For $x = t - 2t^2$ and $y = 24t^4 - 6t^2$, $-1 < t < 3$

(a) Find the points where the parametric system has a vertical tangent line.

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$\frac{dy}{dt} = 1 - 4t \quad t = \frac{1}{4}$

t	x	y
$\frac{1}{4}$	$\frac{1}{8}$	$-\frac{9}{32}$

$\frac{1}{4} - 2\left(\frac{1}{16}\right) = \frac{1}{4} - \frac{1}{8}$ $\quad \frac{24}{256} - 6\left(\frac{1}{16}\right) = -\frac{72}{256} = -\frac{9}{32}$

(b) Find the points where there are horizontal tangent lines.

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$\frac{dy}{dt} = 24(4)t^3 - 12t$

$= 12t(8t^2 - 1)$

$= 12t(2\sqrt{2}t - 1)(2\sqrt{2}t + 1)$

$t = 0 \quad t = \frac{\pm 1}{2\sqrt{2}}$

t	x	y
$\frac{-1}{2\sqrt{2}}$	~~-.35355~~	$\frac{24}{64} - 6\left(\frac{1}{8}\right) = -\frac{24}{64}$
0	0	0
$\frac{1}{2\sqrt{2}}$	.10355	$\frac{24}{64} - 6\left(\frac{1}{8}\right) = -\frac{24}{64}$

(c) Find where x is increasing.

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when $1 - 4t > 0$

$1 > 4t$

$\frac{1}{4} > t$

$t \in (-\infty, \frac{1}{4})$

(d) Find where y is increasing.

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$- \quad + \quad - \quad +$

$\frac{-1}{2\sqrt{2}} \quad 0 \quad \frac{1}{2\sqrt{2}}$

y increasing $t \in \left(-\frac{1}{2\sqrt{2}}, 0\right] \cup \left(\frac{1}{2\sqrt{2}}, +\infty\right)$

(e) Sketch the graph of the system on an x-y coordinate system.

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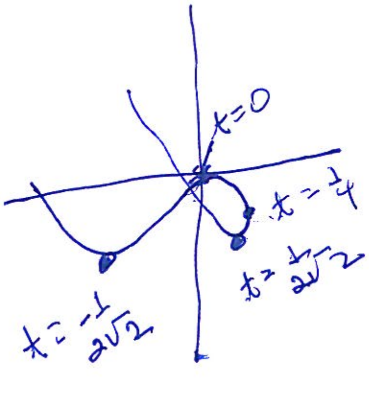
t	x	y	
-1	-3	18	
$\frac{-1}{2\sqrt{2}}$	-.35356	-.375	horizontal
0	0	0	horizontal
$\frac{1}{4}$	.125	.28125	vertical
$\frac{1}{2\sqrt{2}}$	.10356	-.375	horizontal
3	-15	1840	



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XIII. Tell why each series is conditionally convergent, absolutely convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{n^3 + 5}$

divergent since $\lim_{n \to +\infty} \frac{(-1)^n n^3}{n^3+5} \neq 0$

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(b) $\sum_{n=1}^{\infty} (-1)^n 9^n e^{-2n}$

$|r| = \frac{9}{e^2} > 1$

geometric series
and $|r| > 1$
$\therefore$ divergence

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(c) $\sum_{n=1}^{\infty} \frac{(-1)^n \sin(n^4)}{n^3}$

$|a_n| \leq \frac{1}{n^3}$

$\sum \frac{1}{n^3}$ converges $p=3$

$\therefore \sum \left| \frac{(-1)^n \sin(n^4)}{n^3} \right|$ converges

original series converges absolutely

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XIV. Find the radius and interval of convergence for $f(x) = \sum_{n=1}^{\infty} (8x+12)^n 4^{-2n}$.

10 $\lim_{n \to +\infty} |a_n|^{\frac{1}{n}} = \frac{|8x+12|}{16} < 1$ 3

$|8x+12| < 16$

$|x + \frac{3}{2}| < 2$ 2

$R = 2$ 1

$(-\frac{3}{2} - 2, -\frac{3}{2} + 2)$ 1

2 $x = -\frac{3}{2} \pm 2$

$8x = -12 \pm 16$

$\frac{8x+12}{16} = \pm 1$

at either x, series becomes $\sum 1^n$ or $\sum (-1)^n$ neither converges 1 8

$\therefore$ is the interval of convergence

XV. Use a power series to estimate $\int_0^{0.1} \frac{\cos(x^6) - 1}{4x^3} dx$ with an error less than 10^{-25}.

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2 $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$

2 $\frac{\cos x^6 - 1}{4x^3} = \frac{1}{4} \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} x^{12k-3}$

$\int_0^{.1} \frac{\cos x^6 - 1}{4x^3} dx = \frac{1}{4} \sum_{k=1}^{\infty} \frac{(-1)^k}{2k!} \frac{x^{12k-2}}{12k-2} \Big|_0^{.1}$ 2

$= \frac{1}{4} \sum_{k=1} \frac{(-1)^k}{(2k)!} \frac{(.1)^{12k-2}}{12k-2}$ 2

3 $= \frac{1}{4}\Big[\frac{-1}{2} \frac{(.1)^{10}}{10} + \frac{1}{4 \cdot 3 \cdot 2} \frac{(.1)^{22}}{22} - \frac{1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} \frac{(.1)^{34}}{34}$

$1 \sim 10^{-2}$

$\text{Ans} = \frac{1}{4}\Big\{ \frac{-1}{2} \frac{(.1)^{10}}{10} + \frac{1}{4 \cdot 3 \cdot 2} \frac{(.1)^{22}}{22} \Big]$ 4