MAT 162 Final Exam Name ____________________

Fall 2014 Version 1

For full credit, show all work.

I. Calculate the following"

a. $\int x\cos^2(x)\, dx$ $= \int x\left(\frac{1+\cos 2x}{2}\right)dx = \frac{1}{2}\int x\,dx + \frac{1}{2}\int x\cos 2x\,dx$ 3 2

(15) $= \frac{1}{2}\frac{x^2}{2} + \frac{1}{2}\left[x\left(\frac{\sin 2x}{2}\right) - \int \frac{\sin 2x}{2}dx\right]$ 2 2 2

1 $= \boxed{\frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + C}$

check: $\frac{1}{2}x + \frac{1}{4}\sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x = \frac{1}{2}x + \frac{1}{2}x\cos 2x$

$= x\left(\frac{1+\cos 2x}{2}\right) = x\cos^2 x$

(15) b. $\int x^3(1-x^2)^{1/2}dx$ $= \int \sin^3\theta\cos\theta\cos\theta\,d\theta = \int(1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta$ 1 1 1 1

$x = \sin\theta$ 1

$(1-x^2)^{\frac{1}{2}} = (1-\sin^2\theta)^{\frac{1}{2}} = \cos\theta$ 1

$dx = \cos\theta\,d\theta$ 1

$w = \cos\theta$ 1

$dw = -\sin\theta\,d\theta$ 1

$-dw = \sin\theta\,d\theta$ 1

$x^2 + w^2 = \sin^2\theta + \cos^2\theta = 1$

$w = \sqrt{1-x^2}$

$= \int(1-w^2)w^2(-dw)$ 1

$= \int(w^4 - w^2)dw$ 1

$= \frac{1}{5}w^5 - \frac{1}{3}w^3 + C$ 1

$= \boxed{\frac{1}{5}(1-x^2)^{5/2} - \frac{1}{3}(1-x^2)^{3/2} + C}$ 2

II. Tell whether $\int_1^{+\infty}\frac{1+\cos(5x)}{x^3+17}dx$ converges or diverges, and why.

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$\frac{1+\cos(5x)}{x^3+17} \le \frac{2}{x^3}$ 3

$\int_1^{+\infty}\frac{2}{x^3}dx$ converges $\Rightarrow \int_1^{+\infty}\frac{1+\cos(5x)}{x^3+17}dx$ converges. 4

$p = 3$ integral

3

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III. Use the Simpson's rule with n = 6 to estimate $\int_4^7 \frac{x}{1+x^6}\,dx$.

to make 10

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4 4.5 5.5 7

1 · $f(x) = \frac{x}{1+x^6}$

$\Delta x = \frac{7-4}{6} = \frac{3}{6} = \frac{1}{2}$

$\frac{\Delta x}{3}\left(f(4) + 4f(4.5) + 2f(5) + 4f(5.5) + 2f(6) + 4f(6.5) + f(7) \right)$

~~$\frac{1}{6}(f(4) + 4f(4.75) + 2f(5.5) + 4f(6.75) + f(7))$~~

$= \cancel{8.431807886 \times 10^{-4}}$ $8.733158294 \times 10^{-4}$

(Calculator $\int_4^7 \frac{x}{1+x^6}\,dx = 8.723442192 \times 10^{-4}$)

IV. Find the length of the graph of the curve $y = 10 + 4x^{1.5}$, $0 \le x \le 3$.

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$f(x) = 10 + 4x^{1.5}$

$f'(x) = 6x^{.5}$

$\int_0^3 \sqrt{1+36x}\,dx = \frac{2}{3}\cdot\frac{(1+36x)^{3/2}}{36}\Big|_0^3 = \frac{1}{54}(1+36x)^{3/2}\Big|_0^3$

$= \frac{1}{54}\left[109^{3/2} - 1\right] = 21.05543351$

V. Find the centroid of the region in the first and second quadrants bounded by the curves y = (1/7)(x + 50) ,and 100 = x² + y² . Set up the integrals you do not have to solve them.

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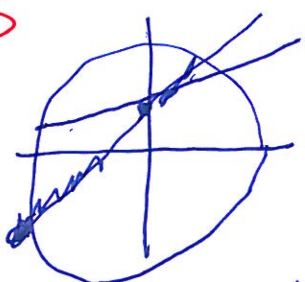

$y = \frac{1}{7}(x+50)$ bottom

$y^2 = \frac{1}{49}(x+50)^2$ top

$x^2 + y^2 = 100$

$\frac{1}{49}(x+50)^2 + x^2 = 100$

$(x+50)^2 + 49x^2 = 4900$

$50x^2 + 100x + 2500 = 4900$

$50x^2 + 100x = 2400$

$x^2 + 2x = 48$

$x^2 + 2x - 48 = 0$

$(x+8)(x-6) = 0$

$x = -8 \quad x = 6$

$Mass = \int_{-8}^{6} \rho\left(\sqrt{10^2 - x^2} - (\tfrac{1}{7})(x+50)\right)dx$

$M_y = \int_{-8}^{6} \rho x\left(\sqrt{10^2 - x^2} - \tfrac{1}{7}(x+50)\right)dx$

$M_x = \int_{-8}^{6} \frac{\rho}{2}\left(10^2 - x^2 - (\tfrac{1}{7})(x+50)^2\right)dx$

$\bar{x} = \frac{M_y}{Mass}$

$\bar{y} = \frac{M_x}{Mass}$

VI. Find k so that f(x) $= \dfrac{k}{x^2+10x}$ if x≥4 and f(x) = 0 if x <4, is a probability density function.

1 1

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$$1 = \int_{4}^{+\infty} \frac{k}{x^2+10x}\,dx = k\int_{4}^{+\infty} \frac{1}{x(x+10)}\,dx = k\int_{4}^{+\infty} \frac{1}{10}\frac{1}{x} - \frac{1}{10}\frac{1}{x+10}\,dx$$

2

$$\frac{1}{x(x+10)} = \frac{A}{x} + \frac{B}{x+10}$$

$1 = A(x+10) + Bx$

$1 = A(10) \Rightarrow A = \frac{1}{10}$; $1 = B(-10)$

$B = -\frac{1}{10}$

$$= k \lim_{b \to +\infty} \frac{1}{10}\ln|x|\Big|_4^b - \frac{1}{10}\ln|x+10|\Big|_4^b$$ 2

$$= k \lim_{b \to +\infty} \frac{1}{10}\ln\left|\frac{x}{x+10}\right|\Big|_4^b$$

$$= k \lim_{b \to +\infty} \frac{1}{10}\ln\left|\frac{b}{b+10}\right| - \frac{1}{10}\ln\frac{4}{14}$$

$$= k\frac{1}{10}\ln\frac{14}{4} \Rightarrow \boxed{k = \frac{10}{\ln\left(\frac{14}{4}\right)}} = \frac{10}{\ln\left(\frac{7}{2}\right)}$$

2 2

VII. Solve completely:

(a) $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+8x}$, y(0) = 1.

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$$\frac{1}{1+y^2}\,dy = \frac{1}{(1+8x)}\,dx$$ 2

$$\arctan y = \frac{1}{8}\ln|1+8x| \; \cancel{x+4x^2} + C$$ 2

$\arctan 1 = 0 + C$ 2

$\frac{\pi}{4} = C$

$$\boxed{y = \tan\left(\cancel{x+4x^2}\ \tfrac{1}{8}\ln|1+8x| + \frac{\pi}{4}\right)}$$ 2 2

(b) $\dfrac{dy}{dx}$ + 8y = 5e^{3x}

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$I(x) = e^{8x}$ 3

$\frac{d}{dx}(ye^{8x}) = 5e^{11x}$ 2

$ye^{8x} = \frac{5}{11}e^{11x} + C$ 3

$$\boxed{y = \frac{5}{11}e^{3x} + Ce^{-8x}}$$ 2

(c) $\dfrac{d^2y}{dx^2}$ -3$\dfrac{dy}{dx}$ -10y = 0.

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3 $r^2 - 3r - 10 = 0$

3 $(r-5)(r+2) = 0$

$r = 5 \quad r = -2$

2

$$\boxed{y = C_1e^{5x} + C_2e^{-2x}}$$

2

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VIII. Use Euler's Method and a stepsize of h = 0.1 to estimate y(.2) where $\frac{dy}{dx} = x+(1+3y)^2$, y(0) = 2.

x	y	$\Delta y = [x + (1+3y)^2] \cdot .1$
1 0	2 1	$[0 + (1+3(2))^2] \cdot .1 = 4.9$ 2
1 .1	6.9 1	$(.1 + (1+3(6.9))^2).1 = 47.099$ 2
1 .2	53.999 1	

IX. A 1000 liter tank is initially filled with brine that contains dissolved salt. A salt solution of .004 kg/l enters the tank at a rate of 50 l/minute; the tank is continuously mixed and a solution drains from the tank at a rate of 70 l/minute. In 20 minutes there is exactly 1 kg of salt in the tank. How much salt was in the tank in the beginning?

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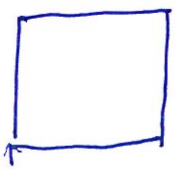

S(t) = amount of salt in tank

S(0) = S_0 unknown

2 $\frac{dS}{dt}$ = rate in − rate out

2 $= 50(.004) - \frac{S \cdot 70}{1000 - 20t}$

2 $= .2 - \frac{3.5}{500 - t} S$

2 $\frac{dS}{dt} + \frac{3.5}{500-t} S = .2$

2 $I(t) = e^{-3.5 \ln(500-t)} = (500-t)^{-3.5}$

2 $\frac{d}{dt}\left(S(t)(500-t)^{-3.5}\right) = .2(500-t)^{-3.5}$

2 $S(t)(500-t)^{-3.5} = .2'(500-t)^{-2.5} + C$

at t = 20

$1(500-20)^{-3.5} = .08(500-20)^{-2.5} + C$ $\quad 30^{-3.5} = .08(30)^{-2.5} + C$ $\quad 1 = .08(30) + C\,30^{3.5}$

~~$480^{-35} = \frac{1}{170}(480)^{-34} + C$~~

2 ~~$480^{-35} - \frac{1}{170}(480)^{-34}$~~ $= C$ $\quad C = \frac{-1.4}{30^{3.5}}$

2 $S(t) = .08(500-t) + C(500-t)^{3.5}$ $\quad \frac{-1.4}{30^{3.5}}$ ~~$(480^{-35} - \frac{1}{170}(480)^{-34})$~~ $500^{3.5} = 4 - 1.4\left(\frac{50}{30}\right)^{3.5} = 4 - 1.4\left(\frac{5}{3}\right)^{3.5}$ $= -4.32755612$

2 $S(0) = .08(500) +$ ~~$(480^{-35} - \frac{1}{170}(480)^{-34})$~~ ~~$\left(\frac{5}{4.8}\right)^{35}$~~

~~$= \frac{50}{17} + \left(\frac{5}{4.8}\right)^{35} - \frac{500}{170}\left(\frac{5}{4.8}\right)^{34} = \frac{50}{17}\left(1 - \left(\frac{5}{4.8}\right)^{34}\right) + \left(\frac{5}{4.8}\right)^{35}$~~

$= -4.12570208$

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X. Find the foci and vertices and sketch the graph of $2x^2 + 4x + y^2 + 16y = 34$.

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$2(x^2 + 2x) + y^2 + 16y = 34$

$2(x^2 + 2x + 1) + y^2 + 16y + 64 = 34 + 2 + 64 = 100$

$\dfrac{(x+1)^2}{50} + \dfrac{(y+8)^2}{100} = 1$

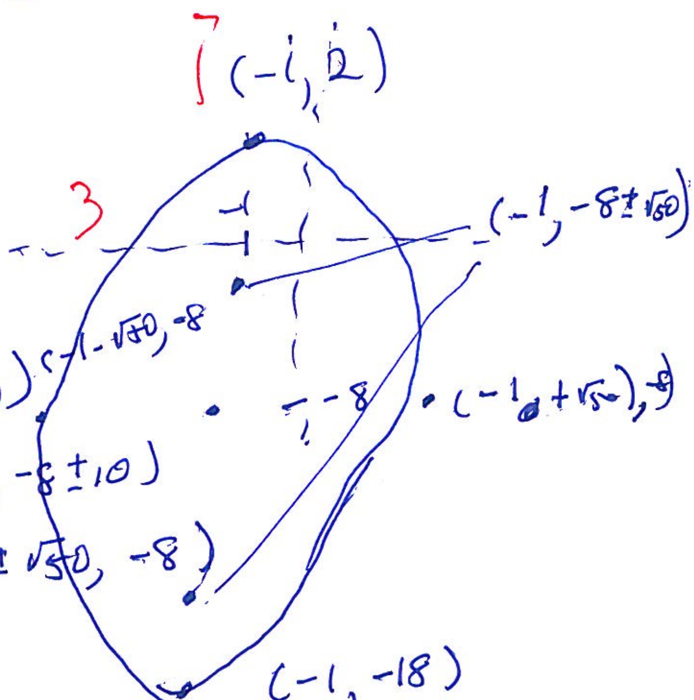
3 center $(-1, -8)$ ellipse

$c^2 = 100 - 50 = 50$

$c = \sqrt{50}$ 2 foci $= (-1, -8 \pm \sqrt{50})$

$a = \sqrt{50}$ 2 major vertices $= (-1, -8 \pm 10)$

$b = 10$ 2 minor " $= (-1 \pm \sqrt{50}, -8)$



XI. Convert r = 6cos(θ)+4sin(θ) into rectangular coordinates and sketch the graph. Find the slope of the tangent line at $\theta = \dfrac{\pi}{2}$.

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$\dfrac{dr}{d\theta} = -6\sin\theta + 4\cos\theta$ $x = r\cos\theta$

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$r^2 = 6r\cos\theta + 4r\sin\theta$

$x^2 + y^2 = 6x + 4y$

$x^2 - 6x + y^2 - 4y = 0$

$x^2 - 6x + 9 + y^2 - 4y + 4 = (\sqrt{13})^2$

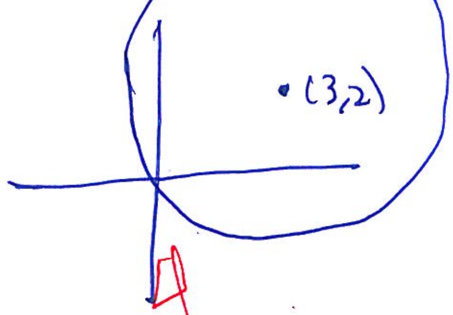
$\dfrac{dx}{d\theta} = \dfrac{dr}{d\theta}\cos\theta - r\sin\theta$ 2

at $\theta = \frac{\pi}{2}$ $= (-6)(0) - 4(1) = -4$

3 circle through the origin centered at (3,2)

$\dfrac{dy}{d\theta} = \dfrac{dr}{d\theta}\sin\theta + r\cos\theta$ 2

at $\theta = \frac{\pi}{2}$ $= (-6)(1) + 4(0) = -6$



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$\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{6}{4} = \boxed{\dfrac{3}{2}}$

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XII. For $x = 4 - t^2$ and $y = 3t^4 - 3t^2$, $1 < t < 3$

(a) Find the points where the parametric system has a vertical tangent line.

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$\frac{dx}{dt} = -2t$ $\quad t = 0$

t	x	y
0	4	0

not in (1,3)
no vertical tan line

(b) Find the points where there are horizontal tangent lines.

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$\frac{dy}{dt} = 12t^3 - 6t$
$= 6t(2t^2 - 1) = 0$
$t^2 = \frac{1}{2}$
$t = \frac{1}{\sqrt{2}}$

t	x	y
$\frac{1}{\sqrt{2}}$	3.5	-.75

$\frac{3}{4} - 3(\frac{1}{2})$
not in (1,3)

(c) Find where x is increasing.

5

$-2t > 0$
$t < 0$

x not increasing in (1,3)

(d) Find where y is increasing.

5

y inc on (1,3)

(e) Sketch the graph of the system on an x-y coordinate system.

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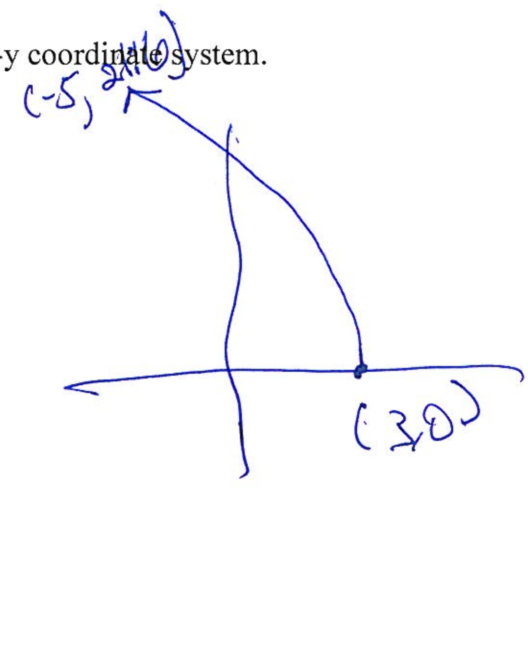
t	x	y
1	3	0
3	-5	216



XIII. Tell why each series is conditionally convergent, absolutely convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{n^3 + 5}$

divergent since $\lim_{n \to +\infty} \frac{(-1)^n n^3}{n^3+5} \neq 0$.

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(b) $\sum_{n=1}^{\infty} (-1)^n 9^n e^{-3n}$

$|r| = \left|\frac{9}{e^3}\right| < 1$ geometric series
and $|r| < 1$
$\therefore$ absolute convergence

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(c) $\sum_{n=1}^{\infty} \frac{(-1)^n \sin(n)}{n^3}$

$|a_n| \leq \frac{1}{n^3}$
$\sum \frac{1}{n^3}$ converges $p = 3$
$\therefore \sum \left|\frac{\sin(n)}{n^3}\right|$ converges
$\therefore$ original series converges absolutely

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XIV. Find the radius and interval of convergence for $f(x) = \sum_{n=1}^{\infty} (6x+12)^n 4^{-2n}$.

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$\lim_{n\to+\infty} |a_n|^{1/n} = \lim_{n\to+\infty} \left|\frac{6x+12}{4^2}\right| = \frac{|6x+12|}{16} < 1$

$|x+2| < \frac{16}{6}$ 2

$R = \frac{16}{6}$ 1

$\left(-2-\frac{16}{6}, -2+\frac{16}{6}\right)$ 1

$-4\frac{2}{3}$ $\frac{2}{3}$

2 $\begin{cases} x = -2 \pm \frac{16}{6} \\ 6x = -12 \pm 16 \\ \frac{6x+12}{16} = \pm 1 \end{cases}$

at $x = -2 \pm \frac{16}{6}$ series becomes $\sum 1^n$

1 & $\sum (-1)^n$

neither converges

is the interval of convergence

XV. Use a power series to estimate $\int_0^{0.1} \frac{\cos(x^5)-1}{4x^3}\,dx$ with an error less than 10^{-25}.

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$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$

$\cos x^5 - 1 = \sum_{k=1}^{\infty} \frac{(-1)^k (x^5)^{2k}}{(2k)!}$

$\frac{\cos x^5 - 1}{4x^3} = \frac{1}{4} \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} x^{10k-3}$

$\int_0^{.1} \frac{\cos x^5 - 1}{4x^3}\,dx = \frac{1}{4} \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!\,(10k-2)} x^{10k-2} \Big|_0^{.1}$

$= \frac{1}{4} \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!\,(10k-2)} (.1)^{10k-2}$

$= \frac{1}{4}\left[\frac{-1}{2(8)}(.1)^8 + \frac{1}{4\cdot 3\cdot 2\,(18)}(.1)^{18} - \frac{1}{(6\cdot 5\cdot 4\cdot 3\cdot 2)(28)}(.1)^{28}\right.$ $\leq 10^{-25}$

$\therefore$ Estimate $\frac{1}{4}\left[\frac{-(.1)^8}{16} + \frac{.1^{18}}{(24)(18)}\right]$