9.3 Solving Separable Differential Equations

Recommended Problems #1-18, 45-48

Defn: A separable D.E. is a first order D.E. in which $\frac{dy}{dx}$ can be written as the product of 2 functions, one involving only x, the other only y:

$$\frac{dy}{dx} = f(x) \cdot g(y) \qquad \text{or} \quad \frac{dy}{dx} = \frac{f(x)}{g(y)}$$

$$\text{or} \quad \frac{dy}{dx} = \frac{g(y)}{f(x)}$$

To solve these: "separate the variables": get everything that involves x (including dx) on one side, + likewise for y + dy on the other. Then integrate on both sides, with respect to x on the x side, with respect to y on the y side.

if $\frac{dy}{dx} = f(x) \cdot g(y)$ if $\frac{dy}{dx} = \frac{f(x)}{g(y)}$

sep vars: $\frac{dy}{g(y)} = f(x)\,dx$ $g(y)\,dy = f(x)\,dx$

integrate $\int \frac{dy}{g(y)} = \int f(x)\,dx$ $\int g(y)\,dy = \int f(x)\,dx$

Result $\underbrace{\text{something}}_{y(x)} = \text{something} + C$ (a family of functions, general solution)

with initial conditions ex: $f(0)=2$, we can find C + we have an exact sol.

Ex 1: (a) Solve the DE $\frac{dy}{dx} = \frac{x^2}{y^2}$

sep. vars: $y^2\,dy = x^2\,dx$

integrate: $\int y^2\,dy = \int x^2\,dx$

$$\frac{y^3}{3} = \frac{x^3}{3} + C$$

solve for y:

$$y^3 = x^3 + \underbrace{3C}_{\substack{\text{still a const} \\ \text{new name: } K}}$$

$$y^3 = x^3 + K$$

$$y = \sqrt[3]{x^3+k} \quad \text{general solution}$$

Initial Value Problem

(b) Find the exact solution if $y(0)=2$.
initial conditions

$$2 = \sqrt[3]{0^3+k} \qquad 2 = \sqrt[3]{k} \qquad k = 8$$

Exact sol: $y = \sqrt[3]{x^3+8}$

Ex 2: Solve the DE: $\frac{dy}{dx} = \frac{6x^2}{2y+\cos y}$

sep vars: $(2y+\cos y)\,dy = 6x^2\,dx$

Integrate: $\int (2y+\cos y)\,dy = \int 6x^2\,dx$

$$\underbrace{y^2+\sin y = 2x^3 + C}_{\text{implicitly-defined function}}$$

(you can't solve for y)

Ex 3: Solve the DE: $y' = x^2 y \quad \left(\frac{dy}{dx} = x^2 y\right)$

sep vars: $\frac{dy}{y} = x^2\,dx$

integrate: $\int \frac{dy}{y} = \int x^2\,dx$

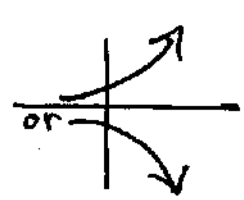
$\ln|y| = \frac{x^3}{3} + C$ use meaning of log to solve for y

$|y| = e^{\left(\frac{x^3}{3}+C\right)}$

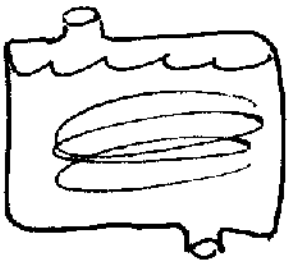
$y = \pm e^{\left(\frac{x^3}{3}+C\right)} = e^{\frac{x^3}{3}} \cdot \underbrace{e^C}_{\text{still a const, new name} = K}$ properties of exponents

$y = \pm K \cdot e^{x^3/3}$

family of exponential functions
(general solution)



Applications of Separable DE's:
pp 598-599 Mixing Problems



t = time (minutes)

$y(t)$ = amount of some substance (in kg, g, whatever) in the liquid that is in the vat, at time t.

Initial conditions:
amount (or %) of the substance in the vat in the beginning ($y(0) = \#$).

You must create the differential equation in order to solve the problem.

Question is usually how much of the substance will there be in the vat at some specified time.

Setting up the differential eqn:

In these problems, <u>rate of flow of liquid</u> is the same going in as coming out, so total volume of liquid stays the same.

But, the concentration of substance in the liquid is different coming in vs. going out.

So, the rates of change of the <u>amt of substance</u> going in & going out are different.

$\frac{dy}{dt}$ = rate of change of amount of substance in vat per unit time

= rate going in of substance − rate going out of substance rate in: $\underbrace{\frac{\text{kg of substance}}{\text{l of liq}}}_{\text{concentration}} \cdot \underbrace{\frac{\#\text{ l of liq}}{\text{min time}}}_{\text{rate of flow of liquid}} = \frac{\text{kg}}{\text{min}}$

rate out: same idea

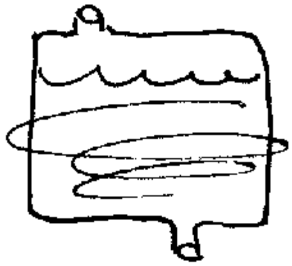
$$\frac{dy}{dt} = \frac{\text{kg going in}}{\text{l}} \cdot \text{rate of flow of liq going in}$$

$$\frac{\text{kg going out}}{\text{l}} \cdot \text{rate of flow going out}$$

and now, for an example:

(we stopped here in the 11 AM class)

Applications: Mixing Problems
pp 598-599



t = time (minutes)
$y(t)$ = amt of some substance that is in the solution (kg, g, whatever) at time t.

Initial Conditions: amt or % in the vat in the beginning $y(0) = \underline{\#}$.
You must create the DE to solve the problem.
Question is often how much will there be in the vat at some other time.

Setup: rate of flow of <u>liquid</u> in + out will be equal
BUT concentration going in + concentration going out are dif.
rate of change of amt of substance going in + going out are dif.

$\frac{dy}{dt}$ = change of amt of substance in the vat $= \frac{kg}{min}$

$$\frac{dy}{dt} = \underset{\text{(of substance)}}{\text{rate going in}} - \underset{\text{(of substance)}}{\text{rate going out}}$$

rate in: concentration coming in $\cdot$ rate of flow of liq

$$\frac{\text{kg subst}}{l \text{ liquid}} \cdot \frac{l \text{ of liquid}}{\text{min}}$$

rate out = concentr. going out $\cdot$ rate of flow of liquid

Ex 6 (p599) Tank w/ 5000 l of water that initially contains 20 k salt in solution.

Pour in brine at 25l/min
↳ [redacted] .03 kg salt/l
Stirring is thorough + continuous.
Liquid is also being drained at the same rate.
Q: How much salt will be in the tank after ½ hr (30 min)?

$y(t)$ = amt of salt in tank (in kg) at time t minutes.
Initial Conditions: $y(0) = 20$

Create DE: $\frac{dy}{dt}$ = rate in − rate out

$$\frac{.03\,kg}{l} \cdot \frac{25\,l}{min} - \frac{y(t)}{5000\,l} \cdot \frac{25\,l}{min}$$

$$\frac{dy}{dt} = .75\,\frac{kg}{min} - \frac{y(t)}{200}\,\frac{kg}{min}$$

$$.75 = \frac{75}{100} = \frac{150}{200}$$

$$\frac{dy}{dt} = \frac{150 - y(t)}{200} = \frac{150 - y}{200}$$

$$\int \frac{dy}{150-y} = \int \frac{1}{200}\,dt \qquad -\ln|150-y| = \frac{1}{200}t + C$$

initial conditions: $y(0) = 20$

$$-\ln|130| = \frac{1}{200}(0) + C \qquad \therefore C = -\ln(130)$$

solution to DE: $-\ln|150-y| = \frac{1}{200}t + C$

solve for y: (10 AM class stopped here)

Finished after 10AM class, to post on web page:

$$-\ln|150-y| = \frac{1}{200}t - \ln 130$$

use meaning to solve for y:

but first mult thru by -1:

$$\ln|150-y| = \frac{-t}{200} + \ln 13$$

now: $\underbrace{|150-y|} = e^{-\frac{t}{200} + \ln 130}$

note: $.03 \times 5000$ is max y could be, when the vat is full of brine,

so $|150-y|$ is $150-y$

so $150 - y = e^{-\frac{t}{200} + \ln 130}$

$$= e^{-\frac{t}{200}} \cdot \underbrace{e^{\ln 130}}_{130}$$

so $150 - y = 130e^{-t/200}$

so $150 - 130e^{-t/200} = y$

Now to answer the Q: $y = 150 - 130e^{-t/200}$

put in $t = 30$ min

$y \approx 38.1$ kg of salt in vat after 1/2 hour.