100 points / 10 points per problem

Test 2
MAT 162

S II, 2012
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Name: Key

Directions: Show all work for partial credit purposes. You may use a graphing calculator. The test is closed book.

10

1. Use Euler's Method to approximate y(2.2) given $\frac{dy}{dx} = 2x - 3y - 4$, and y(2) = 1. Use a stepsize of 0.1 .

guide

x	y	F(x,y)(.1)
2	1	-.3
2.1	.7	-.19
2.2	.51	

$4 - 3 - 4 = -3$

$4.2 - 2.1 - 4 = 2.1 - 4 = -1.9$

2. A. Find the equilibrium solutions to $\frac{dy}{dx} = (y-6)(y+1)^2$.

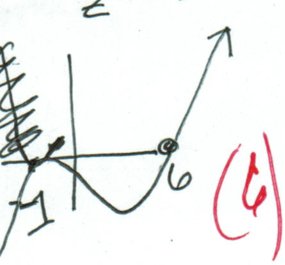
(4)

$y = 6$

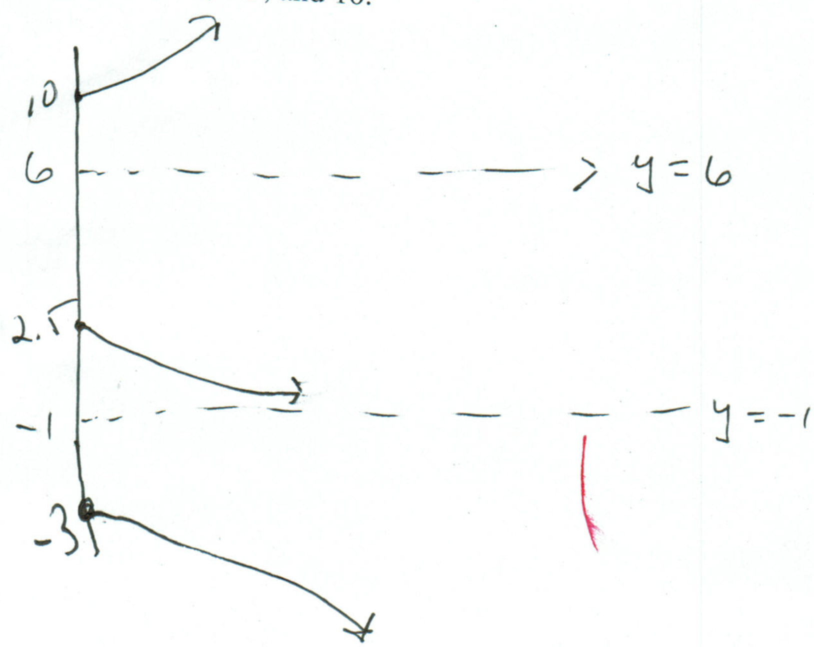
$y = -1$

B. On a single graph, sketch three solutions to $\frac{dy}{dx} = (y-6)(y+1)^2$ for the three differential initial conditions: y(0) = -3 , 2.5, and 10.

$z = (y-6)(y+1)^2$



(6)



3. Find y(x), the solution to $\frac{dy}{dx} = -2x^{-1}y+x+1$, $y(1) = 3$.

(10)

$\frac{dy}{dx} + 2x^{-1}y = x+1$ 2

$\mu = e^{\int 2x^{-1}dx} = e^{2\ln x} = x^2$ 2

$x^2\frac{dy}{dx} + 2xy = x^2(x+1)$ 2

$\frac{d}{dx}(x^2y) = x^3+x^2$ 2

$x^2y = \frac{1}{4}x^4 + \frac{1}{3}x^3 + C$ 1

$3 = \frac{1}{4} + \frac{1}{3} + C$

$3 - \frac{7}{12} = C$

$C = \frac{29}{12}$ 1

$\boxed{y = \frac{1}{4}x^2 + \frac{1}{3}x + Cx^{-2}}$

4. Find y(x), the solution to $\frac{dy}{dx} = x^2(1+y)^2$, $y(0) = \pi/4$.

(10)

3 $\frac{1}{(1+y)^2}dy = x^2dx$

3 $\frac{-1}{1+y} = \frac{1}{3}x^3 + C$

2 $\frac{-1}{1+\frac{\pi}{4}} = C$

$-\frac{1}{1+y} = \frac{1}{3}x^3 - \frac{1}{1+\frac{\pi}{4}}$

$1+y = \frac{1}{\frac{1}{1+\frac{\pi}{4}} - \frac{1}{3}\cancel{x}^3}$

2 $\boxed{y = \frac{1}{\frac{1}{1+\frac{\pi}{4}} - \frac{1}{3}\cancel{x}^3} - 1}$

(10) 5. A tank is filled with 300 liters of contaminated water containing 3 kg of toxins. Pure water is pumped in at a rate of 40 l/min., mixes instantaneously, and then is pumped out 50 l/min. Find y(t) the number of grams of the toxin in the tank t minutes after the rinse begins. Then find the time at which there is .01 kg of toxin present.

$\frac{dy}{dt}$ = rate in − rate out

4 $= 0 - \frac{y}{300-10t} 50$, $y(0) = 3$

1 $\frac{1}{y}\,dy = -\frac{50}{300-10t}\,dt$

1 $\ln y = 5\ln(300-10t) + c$

1 $y = e^c(300-10t)^5$

$3 = A(300)^5 \Rightarrow A = 3(300)^{-5}$

$y(t) = 3(300)^{-5}(300-10t)^5$

1 $y(t) = 3\left(1-\frac{t}{30}\right)^5$

$3\left(1-\frac{t}{30}\right)^5 = .01$

$1-\frac{t}{30} = \left(\frac{.01}{3}\right)^{\frac{1}{5}}$ 1

$\frac{t}{30} = 1-\left(\frac{.01}{3}\right)^{\frac{1}{5}}$

$\boxed{t = 30\left(1-\left(\frac{.01}{3}\right)^{\frac{1}{5}}\right)}$ 1

≈ 20.413

(10) 6. First find the solution to $\frac{d^2y}{dx^2} - 25y = 0$, y(0) = 1, y'(0) = 2.

1 $r^2 - 25 = 0$

1 $(r-5)(r+5) = 0$

3 $y(x) = c_1e^{5x} + c_2e^{-5x}$

$y'(x) = 5c_1e^{5x} - 5c_2e^{-5x}$

1 $1 = y(0) \Rightarrow c_1 + c_2 = 1$

1 $2 = y'(0) \Rightarrow 5c_1 - 5c_2 = 2$

$\Rightarrow$

$5c_1 + 5c_2 = 5$

$5c_1 - 5c_2 = 2$ 1

$10c_1 = 7$

$c_1 = \frac{7}{10}$

$c_2 = \frac{3}{10}$ 1

1 $\boxed{y(x) = \frac{7}{10}e^{5x} + \frac{3}{10}e^{-5x}}$

10 7. Find the value of k so that $f(x) = x^{-12} + kx^{-3}$ is a probability density function on $[1,+\infty)$ and then find the value of the mean for the probability density function.

5 $1 = \int_1^{+\infty} (x^{-12} + kx^{-3})\,dx = \lim_{b \to +\infty} \int_1^b (x^{-12} + kx^{-3})\,dx$ 1 1

$= \lim_{b \to +\infty} \left(\frac{x^{-11}}{-11} + \frac{kx^{-2}}{-2} \right) \Big|_1^b$ 1

$= \lim_{b \to +\infty} \left(\frac{b^{-11}}{-11} + k\frac{b^{-2}}{-2} + \frac{1}{11} + \frac{k}{2} \right)$ 1 1

$= \frac{1}{11} + \frac{k}{2} \Rightarrow \frac{k}{2} = \frac{10}{11} \Rightarrow \boxed{k = \frac{20}{11}}$

1

5 $\mu = \int_1^{+\infty} x(x^{-12} + kx^{-3})\,dx = \int_1^{+\infty} (x^{-11} + kx^{-2})\,dx$ 1

$= \lim_{b \to +\infty} \left(\frac{x^{-10}}{-10} + \frac{kx^{-1}}{-1} \right) \Big|_1^b = \frac{1}{10} + k = \frac{1}{10} + \frac{20}{11} = \boxed{\frac{211}{110}}$

1 1 1

8. For $f(x) = 17 + 6x$, find the length of the curve $y = f(x)$ from $x = 2$ to 8.

1

$s = \int_2^8 \sqrt{1 + (f'(x))^2}\,dx = \int_2^8 \sqrt{1 + 6^2}\,dx = \sqrt{37}\, x \Big|_2^8$

10 1 1 2 2

$= \sqrt{37}(8-2) = \boxed{6\sqrt{37}}$

2

9. Find the area of the surface generated by rotating about the x-axis the graph of y = 2+5x from 0 to π/4.

10

$$S = \int_0^{\pi/4} 2\pi(2+5x)\sqrt{1+5^2}\;dx$$

$$= \int_0^{\pi/4} 2\sqrt{26}\,\pi\,(2+5x)$$

$$= 2\sqrt{26}\,\pi\left(2x+\frac{5}{2}x^2\right)\Big|_0^{\pi/4} \quad 2$$

$$\left.\begin{aligned} &= 2\sqrt{26}\,\pi\left(2\left(\frac{\pi}{4}\right)+\frac{5}{2}\left(\frac{\pi^2}{16}\right)\right) \\ &= 2\sqrt{26}\,\pi\left(\frac{\pi}{2}+\frac{5\pi^2}{32}\right) \end{aligned}\right\} 2$$

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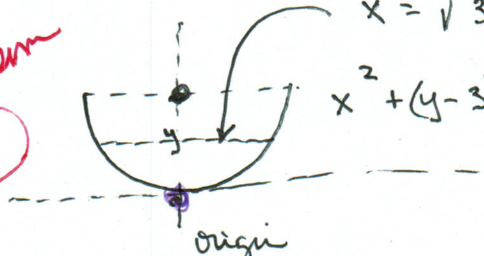
10. A cylindrical barrel of radius 3 feet is lying on its side half-filled with water weighing 62.5 lbs. per cubic foot. Find the hydrostatic force against one end of the barrel.

diagram 2



$$x = \sqrt{3^2-(y-3)^2}$$

$$x^2+(y-3)^2 = 3^2$$

1 width $= 2\sqrt{9-y^2+6y-9}$

$= 2\sqrt{6y-y^2}$

1 Area $= 2\sqrt{6y-y^2}\,dy$ or $2\sqrt{6y-y^2}\,\Delta y$

1 depth $= 3-y$

$$3 \quad \int_0^3 62.5\,(3-y)\,2\sqrt{6y-y^2}\,dy$$

$u = 6y-y^2$

$du = 6-2y$

$\frac{1}{2}du = 3-y$

$$2\left\{\begin{aligned} &= \int_0^9 (62.5)(2)\,u^{\frac{1}{2}}\,\frac{1}{2}\,du \\ &= \int_0^9 62.5\,u^{\frac{1}{2}}\,du = 62.5\left(\frac{2}{3}\right)u^{3/2}\Big|_0^9 \\ &= 62.5\left(\frac{2}{3}\right)9^{3/2} = 62.5(2)(3^2) \\ &= \underline{125(9)\text{ lbs.}} \end{aligned}\right.$$

1125