MAT 162
Summer II 2012

Final Exam

Name Key

For full credit, show all work.

Each page is worth 30-40 points. There are a total of 280 points in this final. Several problems can be done in more than one way. For most only one correct answer is given.

I. Calculate the following"

(15) a. $\int x\cos^2(x)dx = \int x\left(\frac{1+\cos 2x}{2}\right)dx = \frac{1}{2}\int\left[x + x\cos(2x)\right]dx = \frac{1}{2}\left[\frac{x^2}{2} + \int x\cos(2x)dx\right]$

Use parts: $u = x$; $dv = \cos(2x)dx$ to integrate $\int x\cos(2x)dx = uv - \int v\,du$

$du = dx \quad v = \frac{1}{2}\sin(2x)$

$= \frac{1}{2}x\sin(2x) - \int \frac{1}{2}\sin(2x)dx$

$= \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + K.$

$\therefore \int x\cos^2(x)\,dx = \frac{x^2}{4} + \frac{1}{2}\left[\frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x)\right] + C$

$= \boxed{\frac{x^2}{4} + \frac{1}{4}x\sin(2x) + \frac{1}{8}\cos(2x) + C}$

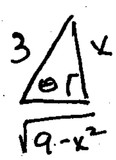
(15) b. $\int x^3(9-x^2)^{3/2}dx = \int 3^3\sin^3\theta\, 3^3\cos^3\theta\, 3\cos\theta\, d\theta = \int 3^7\sin^2\theta\cos^4\theta\sin\theta\, d\theta$

$x = 3\sin\theta$

$9 - x^2 = 9\cos^2\theta$

$(9-x^2)^{\frac{1}{2}} = 3\cos\theta$

$dx = 3\cos\theta\, d\theta$



$\cos\theta = \frac{\sqrt{9-x^2}}{3}$

$= \int 3^7(1-\cos^2\theta)\cos^4\theta(\sin\theta)d\theta \qquad u = \cos\theta = \frac{\sqrt{9-x^2}}{3}$, $du = -\sin\theta\, d\theta$

$= -\int 3^7(1-u^2)u^4du = -3^7\int(u^4-u^6)du$

$= -3^7\left[\frac{1}{5}u^5 - \frac{1}{7}u^7\right] + C$

$= -3^7\left[\frac{1}{5}\frac{(9-x^2)^{5/2}}{3^5} - \frac{1}{7}\frac{(9-x^2)^{7/2}}{3^7}\right] + C$

$= \boxed{\frac{1}{7}(9-x^2)^{7/2} - \frac{9}{5}(9-x^2)^{5/2} + C}$

(40) II. Tell whether $\int_2^{+\infty}\frac{2+\cos(x)}{x^2+2x+17}dx$ converges or diverges, and why.

On $[2, +\infty)$, $\frac{2+\cos x}{x^2+2x+7} \leq \frac{3}{x^2}$.

$\int_2^{+\infty}\frac{3}{x^2}dx$ converges by $p = 2$ function test.

$\therefore$ By Comparison Test, $\int_2^{+\infty}\frac{2+\cos x}{x^2+2x+7}dx$ $\boxed{\text{converges.}}$

/40

III. Use the Simpson's rule with n = 6 to estimate $\int_1^3 \frac{x^2}{1+x^4}\,dx$. [remember midpoint, trapezoid, Simpson's]

(10) Set $f(x) = \frac{x^2}{1+x^4}$; $\Delta x = \frac{3-1}{6} = \frac{1}{3}$; $\frac{\Delta x}{3} = \frac{1}{9}$

$4/3 \quad 5/3 \quad 7/3 \quad 8/3$; $1 \quad 2 \quad 3$

Answer: $\boxed{\frac{1}{9}\left[f(1) + 4f(\tfrac{4}{3}) + 2f(5/3) + 4f(2) + 2f(7/3) + 4f(8/3) + f(3)\right]}$

full credit

$= .523007053\,6$ (Not: $.53445\,7102$ which is calculator value of $\int_1^3 \frac{x^2}{1+x^4}\,dx$)

IV. Find the length of the graph of the curve $y = f(x)$, $0 \le x \le 4$, if $dy/dx = (\cos(x))^{.5}$.

(2)

$$L = \int_0^4 \sqrt{1+\cos x}\,dx = \int_0^4 \sqrt{2}\sqrt{\frac{1+\cos x}{2}}\,dx = \int_0^4 \sqrt{2}\sqrt{\cos^2(\tfrac{x}{2})}\,dx$$

$$= \int_0^4 \sqrt{2}\,\left|\cos(\tfrac{x}{2})\right|dx = \int_0^\pi \sqrt{2}\cos(\tfrac{x}{2})\,dx - \int_\pi^4 \sqrt{2}\cos(\tfrac{x}{2})\,dx$$

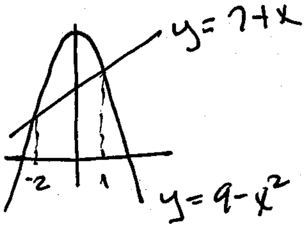
$$= 2\sqrt{2}\sin(\tfrac{x}{2})\Big|_0^\pi - 2\sqrt{2}\sin(\tfrac{x}{2})\Big|_\pi^4$$

$$= 2\sqrt{2} - [2\sqrt{2}\sin(2) - 2\sqrt{2}] = \boxed{4\sqrt{2} - 2\sqrt{2}\sin(2)}$$

$$= 3.084972743$$

V. Find the centroid of the region bounded by the curves $y = 7 + x$, $y = 9 - x^2$.

(13)



$7+x = 9-x^2$

$x^2+x-2=0$

$(x+2)(x-1)=0$

$x=-2$ & $x=1$

$$M = \rho\int_{-2}^1 [9-x^2-(7+x)]\,dx = \rho\int_{-2}^1 (2-x^2-x)\,dx$$

$$= \rho\left[2x - \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right]\Big|_{-2}^1$$

$$= \rho\left[(2-\tfrac{1}{3}-\tfrac{1}{2}) - (-4+\tfrac{8}{3}-2)\right] = \rho\left[8-3-\tfrac{1}{2}\right]$$

$$= 4.5\rho$$

$$M_y = \rho\int_{-2}^1 x\left((9-x^2)-(7+x)\right]dx$$

$$= \rho\int_{-2}^1 (2x - x^3 - x^2)\,dy$$

$$= \rho\left(x^2 - \tfrac{1}{4}x^4 - \tfrac{1}{3}x^3\right)\Big|_{-2}^1$$

$$= \rho\left(1 - \tfrac{1}{4} - \tfrac{1}{3} - (4-4+\tfrac{8}{3})\right)$$

$$= \rho(-2-\tfrac{1}{4}) = -2.25\rho$$

$$\boxed{\bar{x} = \frac{M_y}{M} = \frac{-2.25\rho}{4.5\rho} = -\frac{1}{2}}$$

$$M_x = \rho\int_{-2}^1 \tfrac{1}{2}\left[(f(x))^2 - (g(x))^2\right]dx$$

$$= \frac{\rho}{2}\int_{-2}^1 \left((9-x^2)^2 - (7+x)^2\right)dx = \frac{\rho}{2}\int_{-2}^1 (32 - 14x - 19x^2 + x^4)\,dx$$

$$= \frac{\rho}{2}\left[32x - 7x^2 - \tfrac{19}{3}x^3 + \tfrac{1}{5}x^5\right]\Big|_{-2}^1$$

$$= \frac{\rho}{2}\left[\frac{333}{5}\right] = \rho(33.3)$$ ← used calculator

$$\boxed{\bar{y} = \frac{M_x}{M} = \frac{33.3\rho}{4.5\rho} = 7.4}$$

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VI. Find k so that $f(x) = \dfrac{k}{x^2+x}$ if $x \geq 3$ and $f(x) = 0$ if $x < 3$, is a probability density function.

(10) $\dfrac{1}{x(x+1)} = \dfrac{A}{x} + \dfrac{B}{x+1}$; $1 = A(x+1) + Bx \Rightarrow \dfrac{1}{x(x+1)} = \dfrac{1}{x} - \dfrac{1}{x+1}$

$x=0 \quad A=1$

$x=-1 \quad B=-1$

$1 = \lim_{b\to+\infty} k\int_3^b \left(\frac{1}{x} - \frac{1}{x+1}\right)dx = \lim_{b\to+\infty} k\left[\ln b - \ln(b+1) - (\ln 3 - \ln 4)\right]$

$= k \lim_{b\to+\infty}\left[\ln\left(\frac{b}{b+1}\right) - \ln\left(\frac{3}{4}\right)\right] = k\left(-\ln\frac{3}{4}\right) = k\ln\frac{4}{3} \Rightarrow \boxed{k = \frac{1}{\ln\frac{4}{3}}}$

VII. Solve completely:

(10) (a) $\dfrac{dy}{dx} = \dfrac{y}{(2x^2+1)}$, $y(0) = 2$.

$\int \frac{1}{y}dy = \int \frac{1}{2x^2+1}dx = \frac{1}{\sqrt{2}}\int \frac{\sqrt{2}}{1+(\sqrt{2}x)^2}dx = \frac{1}{\sqrt{2}}\arctan(\sqrt{2}x) + C$

$\therefore \ln y = \frac{1}{\sqrt{2}}\arctan(\sqrt{2}x) + C$

$\ln 2 = 0 + C \Rightarrow C = \ln 2$

$\ln y = \frac{1}{\sqrt{2}}\arctan(\sqrt{2}x) + \ln 2$

$\boxed{y = 2e^{\frac{1}{\sqrt{2}}\arctan(\sqrt{2}x)}}$

(10) (b) $\dfrac{dy}{dx} - 2xy = 5e^{x(x+1)}$

$\mu = e^{\int -2x\,dx} = e^{-x^2}$

$\frac{d}{dx}(e^{-x^2}y) = 5e^{x^2+x}e^{-x^2} = 5e^x$

$e^{-x^2}y = 5e^x + C$

$\boxed{y = 5e^{x^2+x} + Ce^{x^2}}$

(10) (c) $\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 17y = 0$.

$r^2 - 8r + 17 = 0$

$r^2 - 8r + 16 + 1 = 0$

$(r-4)^2 = -1$

$r - 4 = \pm i$

$r = 4 \pm i$

$\boxed{y(x) = c_1e^{4x}\cos x + c_2e^{4x}\sin x}$

/40

VIII. Use Euler's Method and a stepsize of h = 0.1 to estimate y(.2) where $\frac{dy}{dx} = .5x+(1+y^2)$, y(0) = 2.

(10)

x	y	f(x,y)(.1)
0	2	.5
.1	2.5	.73
.2	3.23	

$\boxed{3.23}$

$f(x,y) = .5x + (1+y^2)$

$f(0,2) = (1+2^2) = 5$

$f(.1, 2.5) = .05 + 1 + (6.25) = 7.3$

IX. A 2000 liter tank is initially filled with brine that contains 4 kg of dissolved salt. A salt solution of .003 kg/l enters the tank at a rate of 50 l/minute; the tank is continuously mixed and a solution drains from the tank at a rate of 60 l/minute. How much salt is in the tank at t minutes for t < 200 minutes?

(25)

$S(0) = 4$

$\frac{ds}{dt}$ = rate in − rate out

$= .003(50) - \frac{S(t)}{2000-10t}(60)$

$\frac{ds}{dt} + \frac{6}{200-t} S(t) = .15$

$\mu(t) = e^{\int \frac{6}{200-t}} = e^{-6 \ln(200-t)} = (200-t)^{-6}$

$\frac{d}{dt}\left((200-t)^{-6} S(t)\right) = .15(200-t)^{-6}$

$\therefore (200-t)^{-6} S(t) = \frac{.15}{+5}(200-t)^{-5} + C$

$\therefore S(t) = .03(200-t) + C(200-t)^6$

$4 = S(0) = .03(200) + C(200)^6$

$4 = 6 + C(200)^6$

$\frac{-2}{200^6} = C$

$\therefore S(t) = .03(200-t) - \frac{2}{200^6}(200-t)^6$

$\boxed{S(t) = .03(200-t) - 2(1-.005t)^6}$

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X. Find the foci and vertices and sketch the graph of $-200y^2 + 25x^2 - 250x - 400y = 4575$.

(20) $25(x^2-10x) - 200(y^2+2y) = 4575$

$25(x^2-10x+25) - 200(y^2+2y+1) = 4575 + 625 - 200 = 5000$

$\therefore \frac{(x-5)^2}{200} - \frac{(y+1)^2}{25} = 1$

hyperbola

center $(5,-1)$

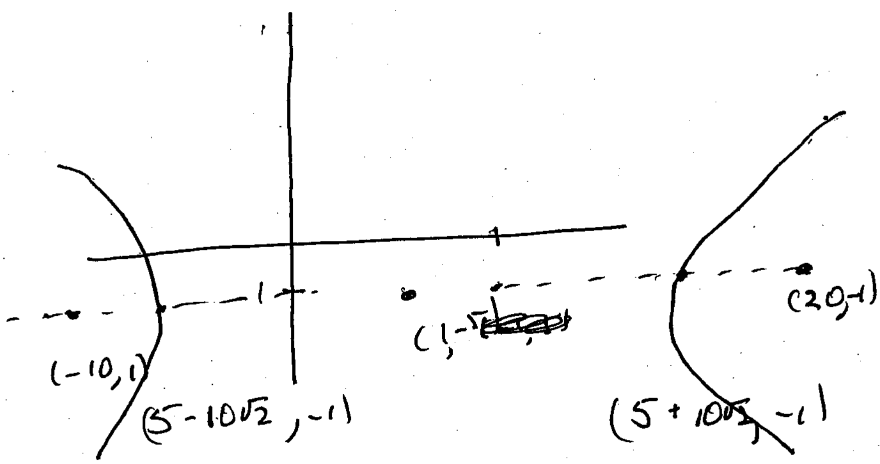
$c^2 = a^2+b^2 = 225$

$c = 15$

$a = 10\sqrt{2}$

vertices $(5 \pm 10\sqrt{2}, -1)$

foci $(5 \pm 15, -1)$



XI. Convert $r = 8\sin(\theta) + 4\cos(\theta)$ into rectangular coordinates and sketch the graph. Find the slope of the tangent line at $\theta = \frac{\pi}{6}$.

(20)

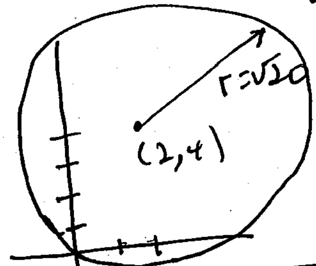
$r^2 = 8r\sin\theta + 4r\cos\theta$

$x^2+y^2 = 8y+4x$

$x^2-4x+4 + y^2-8y+16 = 20$

$(x-2)^2 + (y-4)^2 = 20^2$

circle centered at $(2,4)$ with radius of $\sqrt{20}$



there are two ways to find $\frac{dy}{dx}$ when $\theta = \frac{\pi}{6}$.

Method I $x = r\cos\theta$, $y = r\sin\theta$

$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\sin\frac{\pi}{6} = \frac{1}{2}$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$

$= \frac{(8\cos\theta - 4\sin\theta)\sin\theta + (8\sin\theta + 4\cos\theta)\cos\theta}{(8\cos\theta - 4\sin\theta)\cos\theta - (8\sin\theta + 4\cos\theta)\sin\theta}$

$= \frac{(2\cos\theta - \sin\theta)(\sin\theta) + (2\sin\theta + \cos\theta)\cos\theta}{(2\cos\theta - \sin\theta)(\cos\theta) - (2\sin\theta + \cos\theta)\sin\theta}$

$= \frac{(\sqrt{3} - \frac{1}{2})(\frac{1}{2}) + (1 + \frac{\sqrt{3}}{2})(\frac{\sqrt{3}}{2})}{(\sqrt{3} - \frac{1}{2})\frac{\sqrt{3}}{2} - (1 + \frac{\sqrt{3}}{2})(\frac{1}{2})} = \frac{\sqrt{3} + \frac{1}{2}}{1 - \frac{\sqrt{3}}{2}} = \boxed{\frac{1+2\sqrt{3}}{2-\sqrt{3}}}$

≈ 16.66602539

or Method II: when $\theta = \frac{\pi}{6}$,

$r = 8(\frac{1}{2}) + 4\frac{\sqrt{3}}{2} = 4 + 2\sqrt{3}$

$x = (4+2\sqrt{3})(\frac{\sqrt{3}}{2}) = (2+\sqrt{3})\sqrt{3} = 3 + 2\sqrt{3}$

$y = (4+2\sqrt{3})(\frac{1}{2}) = 2 + \sqrt{3}$

$2(x-2) + 2(y-4)\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2(x-2)}{2(y-4)} = \frac{2-x}{y-4} = \frac{2-3-2\sqrt{3}}{2+\sqrt{3}-4} = \frac{-1-2\sqrt{3}}{-2+\sqrt{3}} = \boxed{\frac{1+2\sqrt{3}}{2-\sqrt{3}}}$

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XII. For $x = t^3 - 3t$ and $y = t^3 - 12t$, $-3 < t < 3$

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(a) Find the points where the parametric system has a vertical tangent line.

$\frac{dy}{dt} = 3t^2 - 3 = 0$

$= 3(t-1)(t+1) = 0$

$t = 1$

$t = -1$

t	x	y
-1	2	11
1	-2	-11

(b) Find the points where there are horizontal tangent lines.

$0 = \frac{dy}{dx} = 3t^2 - 12 = 3(t-2)(t+2)$

$t = \pm 2$

t	x	y
-2	-2	16
2	2	-16

(c) Find where x is increasing.

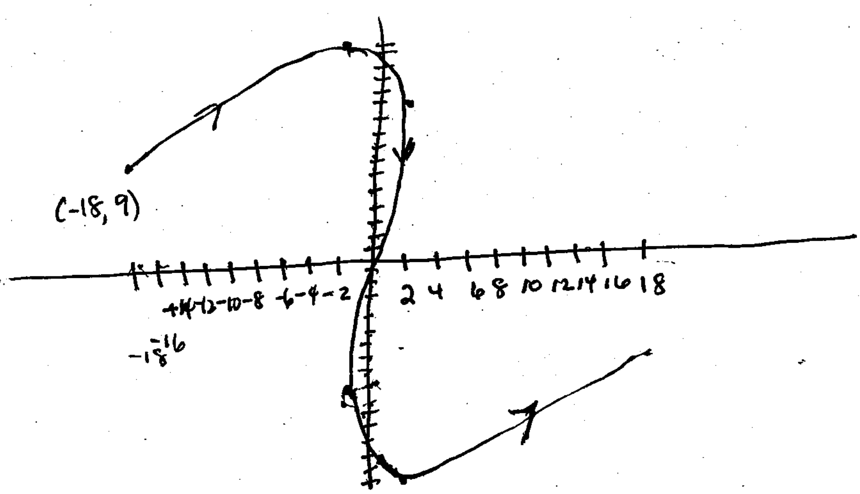
x is increasing for $-3 < t < -1$ and $1 < t < 3$

(d) Find where y is increasing.

y is increasing for $-3 < t < -2$ and $2 < t < 3$

(e) Sketch the graph of the system on an x-y coordinate system.

t	x	y
-3	-18	9
-2	-2	14
-1	2	11
1	-2	-11
2	2	-16
3	18	-9



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XIII. Tell why each series is conditionally convergent, absolutely convergent or divergent.

(10) (a) $\sum_{n=1}^{\infty} \frac{(-1)^n n-1}{n}$ is divergent since $\lim_{n\to+\infty} (-1)\frac{n-1}{n} \neq 0$.

(10) (b) $\sum_{n=1}^{\infty} \frac{(-1)^n (n-2)!}{n!}$ is absolutely convergent since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n (n-2)!}{n!} \right| = \sum_{n=1}^{\infty} \frac{1}{n(n-1)}$$

Compare to $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges $p=2$ series

$\left(\frac{\frac{1}{n(n-1)}}{\frac{1}{n^2}} = \frac{n^2}{n(n-1)} \to 1 \text{ as } n \to +\infty \right)$.

By Limit Comparison $\sum_{n=1}^{\infty} \frac{1}{n(n-1)}$ converges since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

(10) (c) $\sum_{n=1}^{\infty} \frac{(-1)^n (n-1)!}{n!}$ is conditionally convergment

I. It converges by Alt. Series test

(a) $\frac{(n-1)!}{n!} = \frac{1}{n} \to 0$ montonically as $n \to +\infty$

II. (b) Sign's alternate.

$\therefore \sum_{n=1}^{\infty} \frac{(-1)(n-1)!}{n!}$ converges.

But III $\sum_{n=1}^{\infty} \left| \frac{(-1)^n (n-1)!}{n!} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges ($p=1$ series)

$\therefore$ the series is not absolutely convergent.

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XIV. Find the radius and interval of convergence for $f(x) = \sum_{n=1}^{\infty} (2x-5)^n 16^{-.5n}(n+\frac{1}{n})$.

(15) $a_n = \frac{(2x-5)^n}{16^{.5n}}(n+\frac{1}{n}) = \left(\frac{2x-5}{4}\right)^n (n+\frac{1}{n})$

$\therefore \lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty}\left|\left(\frac{2x-5}{4}\right)^{n+1}\frac{n+1+\frac{1}{n+1}}{n+\frac{1}{n}}\frac{4^n}{(2x-5)^n}\right|$

$= \lim_{n\to+\infty}\frac{|2x-5|}{4}\frac{n+1+\frac{1}{n+1}}{n+\frac{1}{n}} = \frac{|2x-5|}{4} < 1$

or $\frac{|x-\frac{5}{2}|}{2} < 1$

$\boxed{\text{radius of convergence} = 2}$.

At $x = \frac{5}{2}+2$, $2x-5=4$ and series is $\sum \frac{4^n}{4^n}(n+\frac{1}{n})$ diverges since $\lim_{n\to+\infty}(n+\frac{1}{n}) \neq 0$

at $x = \frac{5}{2}-2$ series is $\sum_{n=1}^{\infty}(-1)^n\frac{4^n}{4^n}(n+\frac{1}{n})$ which also diverges for the same reason.

$\therefore$ $\boxed{\text{the interval of convergence is } (\frac{1}{2}, \frac{9}{2})}$.

XV. Use a power series to estimate $\int_0^{.01} \frac{\arctan(x^3)}{3}dx$ with an error less than 10^{-12}.

(15) $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$

$\arctan x = \int \frac{1}{1+x^2}dx = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} + C$; $0 = \arctan 0 = 0 + C$

$\therefore \arctan x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} \Rightarrow \frac{1}{3}\arctan(x^3) = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+3}}{2n+1}$

$\int_0^{.01}\arctan(x^3)dx = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+4}}{(2n+1)(6n+4)}\Big|_0^{.01}$

$= \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\frac{(.01)^{6n+4}}{(2n+1)(6n+4)}$

$= \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\frac{10^{-12n-4}}{(2n+1)(6n+4)}$

$n=1$ term is $\frac{1}{3}\frac{10^{-16}}{(3)(10)} < 10^{-12}$

$\therefore$ answer is $\frac{1}{3}\frac{10^{-4}}{4} = \boxed{\frac{1}{12}10^{-8}}$ by Remainder term of Alt. Series.

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