MAT 162 Final Exam Name Key

Summer II 2011

For full credit, show all work.

I. Calculate the following"

a. $\int x\cos(15x)dx$ $= uv - \int v\,du = \frac{1}{15}x\sin(15x) - \int \frac{1}{15}\sin(15x)\,dx$

$= \boxed{\frac{1}{15}x\sin(15x) + \frac{1}{15^2}\cos(15x) + C}$

$u = x \quad dv = \cos(15x)dx$

$\frac{du}{dx} = 1 \quad \frac{dv}{dx} = \cos(15x)$

$du = dx \quad v = \frac{1}{15}\sin(15x)$

b. $\int x^3(x^2+9)^{3/2}dx$ $= \int 3^3\tan^3\theta\; 3^3\sec^3\theta\; 3\sec^2\theta\, d\theta$

$x = 3\tan\theta$

$dx = 3\sec^2\theta\, d\theta$

$x^2+9 = 3^2\tan^2\theta + 9$

$= 9\sec^2\theta$

$\sqrt{x^2+9} = 3\sec\theta$

$w = \sec\theta$

$dw = \sec\theta\tan\theta\, d\theta$

$= \int 3^7\tan^2\theta\sec^4\theta\tan\theta\sec\theta\, d\theta$

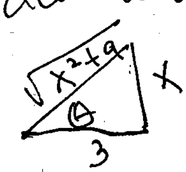
$= \int 3^7(\sec^2\theta - 1)\sec^4\theta\tan\theta\sec\theta\, d\theta$

$= \int 3^7(w^2-1)w^4\,dw$

$= \int 3^7(w^6 - w^4)\,dw$

$= 3^7\left(\frac{1}{7}w^7 - \frac{1}{5}w^5\right) + C$

$= 3^7\left(\frac{1}{7}\sec^7\theta - \frac{1}{5}\sec^5\theta\right) + C$



$\sec\theta = \frac{\sqrt{x^2+9}}{3}$

II. Tell whether $\int_2^{+\infty}\frac{x^5}{x^6-2x-1}dx$ converges or diverges, and why.

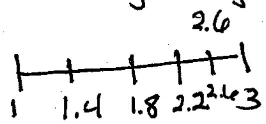
$\frac{x^5}{x^6-2x-1} \geq \frac{x^5}{x^6} = \frac{1}{x}$

$\int_1^{+\infty}\frac{1}{x}$ diverges $(p=1)$

$\therefore$ By Comparison test $\int_1^{+\infty}\frac{x^5}{x^6-2x-1}dx$ diverges.

III. Use the midpoint rule with n = 5 to estimate $\int_1^3 \frac{x^2}{1+x^4}\,dx$. [remember midpoint, trapezoid, Simpson's]

$\Delta x = \frac{3-1}{5} = \frac{2}{5}$



$\frac{2}{5}\left(f(1.2) + f(1.6) + f(2) + f(2.4) + f(2.8)\right)$

where $f(x) = \frac{x^2}{1+x^4}$

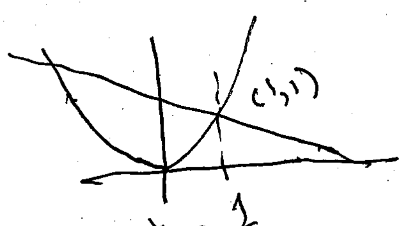
IV. Find the length of the graph of the curve $y = 2x^{1.5}$, $0 \le x \le 3$.

$L = \int_0^3 \sqrt{1+(3x^{.5})^2}\,dx$

$= \int_0^3 \sqrt{1+9x}\,dx = \frac{2}{3}\cdot\frac{1}{9}(1+9x)^{3/2}\Big|_0^3$

$= \frac{2}{27}\left(28^{3/2} - 1^{3/2}\right)$

V. Find the centroid of the region bounded by the curves y = 2 - x , y = x^2, x = 0, and x = 1.



$2 - x = x^2$

$0 = x^2 + x - 2 = (x+2)(x-1) \Rightarrow x = -2 \text{ or } x = 1$

$M_x = \int_0^1 (f(x)-g(x))\left(\frac{f(x)+g(x)}{2}\right) dx$

$= \frac{1}{2}\int_0^1 \rho\left((f(x))^2 - (g(x))^2\right) dx$

$= \frac{\rho}{2}\int_0^1 (2-x)^2 - [x^2]^2\,dx$

$= \frac{\rho}{2}\int_0^1 (4-4x+x^2-x^4)\,dx = \frac{\rho}{2}\left(4x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^1$

$= \frac{\rho}{2}\left(4-2+\frac{1}{3}-\frac{1}{5}\right) = \frac{\rho}{2}\left(2+\frac{2}{15}\right) = \rho\left(1+\frac{1}{15}\right)$

$M = \rho\int_0^1 (2-x-x^2)\,dx = \rho\left(2x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1$

$= \rho\left(2-\frac{1}{2}-\frac{1}{3}\right) = \rho\left(1+\frac{1}{6}\right)$

$M_y = \int_0^1 x\,(f(x)-g(x))\,dx = \int_0^1 \rho x\,(2-x-x^2)\,dx = \rho\left(x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^1$

$= \int_0^1 \rho(2x - x^2 - x^3)\,dx = \rho\frac{5}{12}$

$= \rho\left(1-\frac{1}{3}-\frac{1}{4}\right) =$

$$\boxed{\begin{aligned}\bar{y} &= \frac{M_x}{M} = \frac{1+\frac{1}{15}}{1+\frac{1}{6}} = \frac{32}{35}\\ \bar{x} &= \frac{M_y}{M} = \frac{\rho\left(\frac{5}{12}\right)}{\rho\left(1+\frac{1}{6}\right)} = \frac{5}{14}\end{aligned}}$$

VI. Find k so that f(x) $= \frac{k}{x^2-1}$ if $x \geq 2$ and $f(x) = 0$ if $x < 2$, is a probability density function.

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} \Rightarrow 1 = A(x+1) + B(x-1) \Rightarrow x=1, \quad 1 = A(2) \Rightarrow A = \frac{1}{2}$$

$$x = -1 \quad 1 = B(-2) \Rightarrow B = -\frac{1}{2}$$

$$\int_2^{+\infty} \frac{k}{x^2-1}\,dx = \lim_{b\to+\infty} k\left[\frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x+1)\right]\Big|_2^b$$

$$= \lim_{b\to+\infty} k\left[\frac{1}{2}\ln\left(\frac{x-1}{x+1}\right)\right]_2^b$$

$$= \lim_{b\to+\infty} k\left[\frac{1}{2}\ln\left(\frac{b-1}{b+1}\right) - \frac{1}{2}\ln\left(\frac{1}{2+1}\right)\right] = \frac{k}{2}\ln 3 = 1 \quad \therefore \boxed{k = \frac{2}{\ln 3}}$$

VII. Solve completely:

(a) $\frac{dy}{dx} = \frac{4x}{(2x^2+1)y}$, $y(0) = 2$.

$$\int y\,dy = \int \frac{4x}{2x^2+1}\,dx$$

$$\frac{1}{2}y^2 = \ln(2x^2+1) + C$$

$$\frac{1}{2}(2^2) = \ln 1 + C \Rightarrow C = 2$$

$$\frac{1}{2}y^2 = 2 + \ln(2x^2+1)$$

$$y^2 = 4 + 2\ln(2x^2+1)$$

$$y = \sqrt{4 + 2\ln(2x^2+1)}$$

(b) $\frac{dy}{dx} - 2xy = 5e^{-x}$

$$\mu = e^{\int -2x\,dx} = e^{-x^2}$$

$$\frac{d}{dx}(e^{-x^2}y) = 5e^{-x-x^2}$$

$$e^{-x^2}y = \int 5e^{-z-z^2}\,dz$$ this is not integrable.

$$\boxed{y = e^{x^2}\int 5e^{-z-z^2}\,dz}$$

(c) $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$.

$$r^2 - 8r + 15 = 0$$

$$(r-3)(r-5) = 0$$

$$\boxed{y(x) = c_1 e^{3x} + c_2 e^{5x}}$$

VIII. Use Euler's Method and a stepsize of h = 0.1 to estimate y(.2) where $\frac{dy}{dx} = .5x(1+y^2)$, y(0) = 2.

x	y	$f(x,y)\cdot .1$
0	2	$.5(0)(1+2^2)(.1) = 0$
.1	2	$.5(.1)(1+2^2)(.1) = .05(5)(.1) = ~~.25~~\ .025$
.2	~~2.025~~ 2.025	

$S(0) = 4$ kg

$\frac{ds}{dt}$ = rate in − rate out

$= (.003)(50) - \frac{S(t)}{2000}(50)$

$\frac{ds}{dt} + \frac{1}{40}S(t) = (.003)(50) = .15$

$\mu(t) = e^{\frac{1}{40}t}$

$\frac{d}{dt}\left(e^{\frac{1}{40}t} s\right) = .15 e^{\frac{1}{40}t}$

$e^{\frac{1}{40}t} S = 40(.15) e^{\frac{1}{40}t} + C$

$S = 40(.15) + C e^{-\frac{1}{40}t}$

$S = 6 + C e^{-\frac{1}{40}t}$

$4 = S(0) = 6 + C \Rightarrow C = -2$

$\boxed{S(t) = 6 - 2e^{-\frac{1}{40}t}}$

X. Find the foci and vertices and sketch the graph of $8y^2 + x^2 - 10x + 64y = 47$.

$8(y^2+8y)+x^2-10x=47$

$8(y^2+8y+16)+x^2-10x+25 = 128+25+47 = 200$

$\frac{(y+4)^2}{25} + \frac{(x-25)^2}{200} = 1$

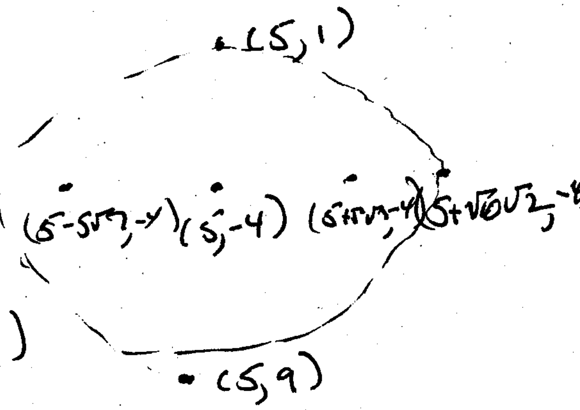
$\frac{(x-5)^2}{(10\sqrt{2})^2} + \frac{(y+4)^2}{5^2} = 1$

$c^2 = 200-25 = 175 = 7(25)$

$c = 5\sqrt{7}$ foci $(5 \pm 5\sqrt{7}, -4)$

vertices $(5 \pm 10\sqrt{2}, -4)$

$(5, -4 \pm 5)$



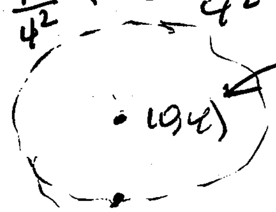
XI. Convert r = 8 sin(θ) into rectangular coordinates and sketch the graph. Find the slope of the tangent line at $\theta = \frac{\pi}{6}$.

$r^2 = 8r\sin\theta$

$x^2+y^2 = 8y$

$x^2+y^2-8y+16 = 16$

$\frac{x^2}{4^2} + \frac{(y-4)^2}{4^2} = 1$



circle of radius 4 centered at (0,4)

$x = r\cos\theta$

$y = r\sin\theta$

$\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$

$= \frac{8\cos\theta\sin\theta + 8\sin\theta\cos\theta}{8\cos\theta\cos\theta - 8\sin\theta\sin\theta}$

$= \frac{8\frac{\sqrt{3}}{2}\left(\frac{1}{2}\right) + 8\frac{\sqrt{3}}{2}\left(\frac{1}{2}\right)}{8\left(\frac{\sqrt{3}}{2}\right)^2 - 8\left(\frac{1}{2}\right)^2}$

$= \frac{16\sqrt{3}}{8(3)-8}$

$= \frac{2\sqrt{3}}{3-1} = \boxed{\sqrt{3}}$

XII. For $x = 12 - t^3$ and $y = t^3 - 27t$, $-4 < t < 4$

$- 27 + 27(3)$

$12 - 27$

$27 - 27(3)$

(a) Find the points where the parametric system has a vertical tangent line.

$\frac{dx}{dt} = 0 = -3t^2 \Rightarrow t = 0 \qquad x = 12,\ y = 0$

(b) Find the points where there are horizontal tangent lines.

$\frac{dy}{dt} = 0 = 3t^2 - 27 = 3(t^2 - 9) = 3(t-3)(t+3)$

t	x	y
-3	39	54
3	-15	-54

(c) Find where x is increasing.

$0 \le \frac{dx}{dt} = -3t^2 \qquad$ x is never increasing ~~decreasing~~

(d) Find where y is increasing.

$0 \le \frac{dy}{dt} = 3(t-3)(t+3)$

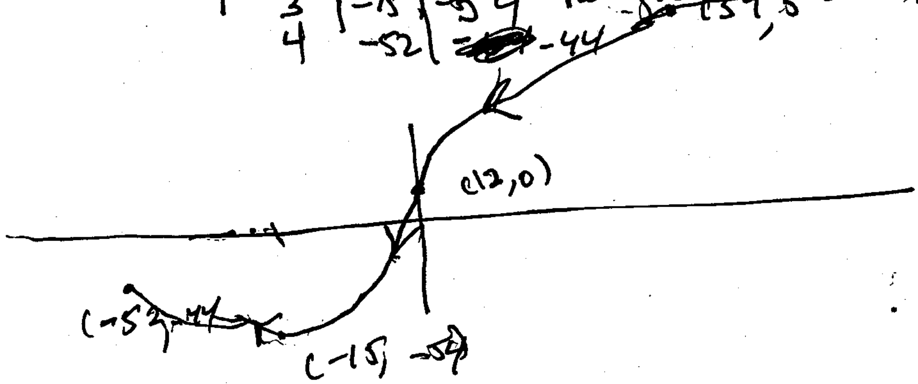
-4 -3 3 4

y is increasing on (-4, -3) and (3, 4)

(e) Sketch the graph of the system on an x-y coordinate system.

t	x	y	
-4	76	42	
-3	39	54	horizontal tan line
0	12	0	vertical tan line
3	-15	~~54~~ -54	horizontal tan line
4	-52	-44	

(39, 54) 76, 42



-64 $\quad 27 \times 4 = 108$, $108 - 64$

-64, 12

-128, 64

$64 - 108$