MAT 162 Test 4 Name Key

Fall 2017

For full credit, show all work.

1. Tell why each series is conditionally convergent, absolutely convergent or divergent.

(a) $\sum_{n=1}^{\infty}\frac{(-1)^n(n-1)}{4n^3-2n+1}$

12

$\sum_{n=1}^{\infty} \frac{n-1}{4n^3-2n+1}$ is convergent thus $\sum_{n=1}^{\infty} \frac{(-1)^n(n-1)}{4n^3-2n+1}$ is absolutely convergent 6

Why? let $a_n = \frac{n-1}{4n^3-2n+1}$ and $b_n = \frac{1}{n^2}$ 2

then $\lim_{n\to+\infty} \frac{a_n}{b_n} = \lim_{n\to+\infty} \frac{n^2(n-1)}{4n^3-2n+1} = \frac{1}{4}$ 2

2

By Limit Comparison, $\sum a_n$ converges since $\sum b_n$ converges (p series $=2$)

(b) $\sum_{n=1}^{\infty}(-1)^n\frac{n}{6n^2+7}$

12

$\sum_{n=1}^{\infty} \frac{n}{6n^2+7}$ is divergent (let $a_n = \frac{n}{6n^2+7}$ $b_n = \frac{1}{n}$ $\frac{a_n}{b_n} = \frac{n^2}{6n^2+7} \to \frac{1}{6}$

$\sum a_n$ diverges since $\sum b_n$ diverges

but $\sum_{n=1}^{\infty} (-1)^n \frac{n}{6n^2+7}$ is convergent by Alt. Series Test

$a_n > 0$, a_n is decreasing and $a_n \to 0$

$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{n}{6n^2+7}$ is conditionally convergent 6

(c) $\sum_{n=1}^{\infty}\frac{(-9)^{n+2}(n+2)}{3^{2n+1}(n+7)}$

12

$\lim_{n\to+\infty} \frac{-9^{n+2}}{3^{2n+1}} = \lim_{n\to+\infty} (-1)^n \frac{9^2}{3} \neq 0$ 6

$\therefore \sum_{n=1}^{\infty} \frac{(9)^{n+2}(n+2)}{3^{2n+1}(n+7)}$ diverges 6

$\lim_{n\to+\infty} \frac{(-1)^{n+2}\, 9^{n+2}(n+2)}{9^n \cdot 3\,(n+7)} = \lim_{n\to+\infty} (1)^{n+2} \frac{9^2}{3} \frac{(n+2)}{n+7} \neq 0$

2. Find the radius and interval of convergence for $f(x) = \sum^{\infty} (x-7)^{n/3}(2+5/n)^{-n}$.

(13)

$$\lim_{n\to+\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to+\infty} \frac{|x-7|^{\frac{1}{3}}}{2+\frac{5}{n}} = \frac{|x-7|^{\frac{1}{3}}}{2} < 1 \quad 4$$

$|x-7|^{\frac{1}{3}} < 2$

$|x-7| < 8$ 2

$R = 8$

$-8 < x-7 < 8$

2 $-1 < x < 15$

$\therefore$ interval is $(-1, 15)$ 1

at $x = 15$ $\sum_{n=1} \frac{8^{n/3}}{(2+\frac{5}{n})^n} = \sum \frac{2^n}{(2+\frac{5}{n})^n}$ 2

$\lim_{n\to+\infty} \frac{2^n}{(2+\frac{5}{n})^n} = \lim_{n\to+\infty} \frac{1}{(1+\frac{5}{2n})^n} = e^{\frac{1}{5/2}} \neq 0$

$\therefore$ diverges at $x = 15$

at $x = -1$ $\sum_{n=1} \frac{(-2)^n}{2+5/n}$ $\lim_{n\to+\infty} (-1)^n \frac{1}{(1+\frac{5}{2n})^n} \neq 0$ 2

diverges at $x=-1$

3. Use a power series to estimate $\int_0^{0.1} x\cos(x^4)dx$ with an error less than 10^{-6}.

(13)

$$\cos x = \sum_{n=0} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow x\cos(x^4) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+1}}{(2n)!} \quad 3$$

$$\int_0^{.1} \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+1}}{(2n)!}\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+2}}{(2n)!\,(8n+2)} \Big|_0^{.1} \quad 3 \quad = \sum_{n=0}^{\infty} \frac{(-1)^n (.1)^{8n+2}}{(2n)!\,(8n+2)} \quad 1$$

$$= \frac{(.1)^2}{2} - \frac{(.1)^{10}}{2(10)}$$

$< 10^{-6}$

(2)

$\boxed{\frac{(.1)^2}{2} \text{ is the answer by Remainder Test for Alt. series}}$ 1

4. Use the fact that the following series is a telescoping series to calculate $\sum_{n=3}^{\infty} \frac{1}{n^2+2n}$ exactly.

(13)

$$\frac{1}{n^2+2n} = \frac{A}{n} + \frac{B}{n+2} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right) \quad 4$$

$1 = A(n+2) + Bn$

$n=0 \quad 1 = A(2) \Rightarrow A = \frac{1}{2}$ 4

$n=-2 \quad 1 = B(-2) \Rightarrow B = -\frac{1}{2}$

4

$$\sum_{n=3}^{+\infty} \frac{1}{n^2+2n} = \frac{1}{2}\left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \frac{1}{2}\left(\frac{1}{4} - \cancel{\frac{1}{6}}\right) + \frac{1}{2}\left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) + \frac{1}{2}\left(\cancel{\frac{1}{6}} - \frac{1}{8}\right)$$

$$+ \frac{1}{2}\left(\cancel{\frac{1}{7}} - \frac{1}{9}\right) + \cdots$$

$$= \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4}\right)$$

$$= \frac{1}{2}\left(\frac{7}{12}\right) = \boxed{\frac{7}{24}} \quad 1$$

5. Calculate $\sum_{n=0}^{\infty} \frac{3^{-2n} 2^{n+2}}{5^{2n}}$ exactly.

(12)

$$= 2^2 \sum_{n=0}^{\infty} \frac{2^n}{(15^2)^n} = 2^2 \sum_{n=0}^{\infty} \left(\frac{2}{225}\right)^n$$

2 2 4

$$= 4\left(\frac{1}{1-\frac{2}{225}}\right) = 4 \frac{1}{\frac{223}{225}} = 4 \frac{225}{223} = \frac{900}{223}$$

1 1

6. Use the integral test to determine the number of terms in the partial sum for $\sum_{n=1}^{\infty} \frac{1}{n^{14}}$ that will estimate the infinite series with an error less than 10^{-11}.

(13)

$$f(x) = \frac{1}{x^{14}}$$ 2

$$R_n \le \int_n^{+\infty} f(x)\,dx = \lim_{b \to +\infty} \frac{x^{-13}}{-13}\Big|_n^b = \frac{1}{13n^{13}}$$

3 3 2

$$\frac{1}{13n^{13}} < 10^{-11}$$ 2

$$\frac{10^{11}}{13} < n^{13}$$ 2

$$\left(\frac{10^{11}}{13}\right)^{\frac{1}{13}} < n .$$ 1

$6 \le n$

MAT 162 Test 4 Name Key

Fall 2017 Version 2

For full credit, show all work.

1. Tell why each series is conditionally convergent, absolutely convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n (n-1)}{4n^3 - 2n + 1}$

12

$\sum_{n=1}^{\infty} \frac{n-1}{4n^3-2n+1}$ is convergent then $\sum_{n=1}^{\infty} \frac{(-1)^n (n-1)}{4n^3-2n+1}$ is absolutely convergent 6

why? let $a_n = \frac{n-1}{4n^3-2n+1}$ $b_n = \frac{1}{n^2}$ then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2(n-1)}{4n^3-2n+1} = \frac{1}{4}$ 2 2

By Limit Comparison, $\sum a_n$ converges since $\sum b_n$ converges (p series $= 2$) 2

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{6n^2+7}$

12

$\sum_{n=1}^{\infty} \frac{n}{6n^2+7}$ is divergent $\left(\frac{n}{6n^2+7} = \frac{1}{6n+\frac{7}{n}} > \frac{1}{6n+7n} = \frac{1}{13n} \right.$ 2

$\frac{1}{13}\sum \frac{1}{n}$ diverges $(p=1)$ forcing $\sum \frac{n}{6n^2+7}$ to diverge

But if $a_n = \frac{n}{6n^2+7}$ then $a_n > 0$, $a_n \to 0$ and a_n is decreasing

$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{n}{6n^2+7}$ converges by Alt. Series Test 2

$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{n}{6n^2+7}$ converges conditionally. 6

(c) $\sum_{n=1}^{\infty} \frac{(-9)^{n+2}(n+2)}{4^{2n+1}(n+7)}$

12

$\left| \frac{(-9)^{n+2}}{4^{2n+1}} \frac{n+2}{n+7} \right| = \left| \frac{9^2}{4} \left(\frac{9}{16}\right)^n \frac{1}{1+\frac{5}{n+2}} \right| < \frac{9^2}{4}\left(\frac{9}{16}\right)^n$ 4

$\sum_{n=1}^{\infty} \frac{9^2}{4}\left(\frac{9}{16}\right)^n$ converges by Geometric Series Test $r = \frac{9}{16}$. 4

$\therefore \sum_{n=1}^{\infty} \frac{(-9)^{n+2}(n+2)}{4^{2n+1}(n+7)}$ converges absolutely. 4

by comparison

2. Find the radius and interval of convergence for $f(x) = \sum_{n=1}^{\infty} (x-7)^{n/3}(2+5/n)^{-n}$.

3. Use a power series to estimate $\int_0^{0.1} x\cos(x^5)\,dx$ with an error less than 10^{-6}.

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow x\cos(x^5) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{10n+1}}{(2n)!} \quad 3$$

$$\int_0^{.1} \sum_{n=0}^{\infty} \frac{(-1)^n x^{10n+1}}{(2n)!}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{10n+2}}{(2n)!\,(10n+2)}\Big|_0^{.1} = \sum_{n=0}^{\infty} \frac{(-1)^n (.1)^{10n+2}}{(2n)!\,(10n+2)} \quad 3$$

$$= \frac{(.1)^2}{2} - \underbrace{\frac{(.1)^{12}}{2(12)}}_{< 10^{-6}} + \cdots \quad 1$$

$\therefore \frac{(.1)^2}{2}$ is the answer by Remainder Test for Alt. Series 1

4. Use the fact that the following series is a telescoping series to calculate $\sum_{n=3}^{\infty} \frac{1}{n^2+2n}$ exactly.

5. Calculate $\sum_{n=0}^{\infty} \frac{3^{-2n}7^{n+2}}{5^{2n}}$ exactly.

$= 7^2 \sum_{n=0}^{\infty} \left(\frac{7}{15^2}\right)^n = 7^2 \sum_{n=0}^{\infty} \left(\frac{7}{225}\right)^n$ 2 4

2

$= 7^2 \left(\frac{1}{1-\frac{7}{225}}\right)$!

7 1 1

6. Use the integral test to determine the number of terms in the partial sum for $\sum_{n=1}^{\infty} \frac{1}{n^{13}}$ that will estimate the infinite series with an error less than 10^{-11}.

$f(x) = \frac{1}{x^{13}}$

$R_n \leq \int_n^{+\infty} f(x)\,dx = \lim_{b \to +\infty} \frac{x^{-12}}{-12}\Big|_n^b = \frac{1}{12n^{12}}$

3 3 2

$\frac{1}{12n^{12}} < 10^{-11}$ 2

$\frac{10^{11}}{12} < n^{12}$ 2

$\left(\frac{10^{11}}{12}\right)^{\frac{1}{12}} < n$ 1

$7 \leq n$