Homework

Turn in by ~~5:00 PM~~ ~~Monday~~ Tuesday classtime

Graded Homework Assignment.

1. (a) Calculate the power series for $\sin x$ centered at $a=0$.

(b) " " " " " " " " $a=\frac{\pi}{2}$.

(c) " " " " " $\cos x$ " " $a=0$

(d) Use the results of (a), (b) + (c) to prove $\cos(x-\frac{\pi}{2})=\sin x$

(e) Using (a), find n so that $T_n(1.6)$ estimates $\sin(1.6)$ with error less than 10^{-6}

(f) Using (b), " " " " $T_n(1.6)$ " $\sin(1.6)$ with error less than 10^{-6}

(g) Find an interval centered at $a=\frac{\pi}{2}$ such that the first three nonzero terms of (b) estimates $\sin x$ with an error less than 10^{-4}.

(h) Find n so that $T_n(x)$ estimates $\sin x$ with an error less than 10^{-5} on $[1.4, 1.8]$

2. Use 1(a) to calculate the first four nonzero terms for $\frac{\sin x}{1+x^2}$.

3. Estimate $\int_0^1 \frac{1-e^{-x^4}}{x^2}$ with an error less than 10^{-4}!

4. True or false: $\int_0^1 \frac{1-e^{-x^4}}{x^2}\,dx = \int_0^1 \frac{1-e^{-x^4}}{x^3}\,dx$.

Why?

1. (g) $\sin x = \underbrace{1 - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{1}{4!}(x-\frac{\pi}{2})^4}_{\text{first three nonzero terms}} - \underbrace{\frac{1}{6!}(x-\frac{\pi}{2})^6}_{\text{error bound for first three nonzero terms}} + \cdots$

$$\left|\frac{1}{6!}\left(x-\frac{\pi}{2}\right)^6\right| < 10^{-4}$$

$$\left|x-\frac{\pi}{2}\right|^6 < 6!\,(10^{-4})$$

$$\left|x-\frac{\pi}{2}\right| < \left[6!\,(10^{-4})\right]^{\frac{1}{6}} = .6449936159$$

$$\frac{\pi}{2} - .649936159 < x < \frac{\pi}{2} + .649936159$$

(h) on $[1.4, 1.8]$, $\left|x-\frac{\pi}{2}\right| < \left|1.8-\frac{\pi}{2}\right| = .2292036732$

$\therefore$ When is $\frac{1}{(2n)!}\left|x-\frac{\pi}{2}\right|^{2n} < \frac{1}{(2n)!}\left|1.8-\frac{\pi}{2}\right|^{2n} < 10^{-5}$

for $n=3$, $\frac{1}{(2n)!}(1.8-\frac{\pi}{2})^{2n} = 2.01\times 10^{-7} < 10^{-5}$

$\therefore$ ~~sin x~~ $1 - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{1}{4!}(x-\frac{\pi}{2})^4$ estimates $\sin x$ with an error less than 10^{-5} on $[1.4, 1.8]$.

2. $$\frac{\sin x}{1+x^2} = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right]\left[\sum_{n=0}^{\infty}(-x^2)^n\right]$$

$$= \left(x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)\left(1 - x^2 + x^4 - x^6 + x^8 - \cdots\right)$$

$$= x + \left(-1-\frac{1}{6}\right)x^3 + \left(\frac{1}{5!}+\frac{1}{6}+1\right)x^5 + \left(-\frac{1}{7!}-\frac{1}{5!}-\frac{1}{6}+1\right)x^7 + \cdots$$

3. $$\int_0^1 \frac{1-e^{x^4}}{x^2}\,dx = \int_0^1 -\sum_{n=1}^{\infty}\frac{(-1)^n x^{4n-2}}{n!}\,dx = -\sum_{n=1}^{\infty}\frac{(-1)^n x^{4n-1}}{n!\,(4n-1)}\Bigg|_0^1$$

$$= -\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{1}{4n-1} = \frac{1}{1!(3)} - \frac{1}{2!(7)} + \frac{1}{3!(11)} - \frac{1}{4!(15)} + \frac{1}{5!(19)} - \underbrace{\frac{1}{6!(23)}}_{<10^{-4}} + \cdots$$

$\therefore$ solution is $\boxed{\frac{1}{(3)} - \frac{1}{2(7)} + \frac{1}{6(11)} - \frac{1}{4!(15)} + \frac{1}{5!(19)}}$

Solutions

	$a=0$	$a=\frac{\pi}{2}$
$f(x) = \sin x$	0	1
$f'(x) = \cos x$	1	0
$f''(x) = -\sin x$	0	-1
$f'''(x) = -\cos x$	-1	0

1. (a) $f(x) = \sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0 + \frac{1}{1!}x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + 0 - \frac{1}{7!}x^7 + \cdots$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}$$

(b) $f(x) = \sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(\frac{\pi}{2})}{n!} (x-\frac{\pi}{2})^n = 1 + 0 - \frac{(x-\frac{\pi}{2})^2}{2!} + 0 + \frac{1}{4!}(x-\frac{\pi}{2})^4 + 0 - \frac{1}{6!}(x-\frac{\pi}{2})^6 + \cdots$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(x - \frac{\pi}{2}\right)^{2n}$$

(c) $f(x) = \cos x = 1 + 0 - \frac{x^2}{2!} + 0 + \frac{1}{4!}x^4 + 0 - \frac{1}{6!}x^6 + \cdots$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} (x^{2n})$$

(d) from (c), $\cos(x - \frac{\pi}{2}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (x - \frac{\pi}{2})^{2n} = \sin x$ from (b)

(e) in (a) $\sin 1.6 = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} 1.6^{2n+1}$

$$= 1.6 - \frac{1}{3!}(1.6)^3 - \frac{1}{5!}(1.6)^5 + \frac{1}{7!}(1.6)^7 - \frac{1}{9!}(1.6)^9$$

4.4×10^{-6} $\quad 7.3 \times 10^{-8}$

Using the Remainder Estimate for an alternating series since $\frac{1}{9!}(1.6)^9 < 10^{-6}$,

$\sum_{n=0}^{3} (-1)^n \frac{1}{(2n+1)!} 1.6^{2n+1}$ estimates $\sin 1.6$ with error less than 10^{-6}

(f) in (b) $\sin 1.6 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (1.6 - \frac{\pi}{2})^{2n}$ $\quad 1 - \frac{(1.6-\frac{\pi}{2})^2}{2!} + \frac{(1.6-\frac{\pi}{2})^4}{4!}$

Since $\frac{(1.6-\frac{\pi}{2})^4}{4!} = 3 \times 10^{-8} < 10^{-6}$, $\quad 1 - \frac{(1.6-\frac{\pi}{2})^2}{2!}$ estimates $\sin 1.6$ with error less than 10^{-6}.

4. From (3)

$$\int_0^1 \frac{1-e^{-x^4}}{x^2}\,dx = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{x^{4n-1}}{4n-1}\Bigg|_0^1$$

$$= -\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{1}{4n-1}.$$

Similarly, $\displaystyle\int_0^1 \frac{1-e^{-x^4}}{x^3}\,dx = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{x^{4n-2}}{4n-2}\Bigg|_0^1$

$$= -\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{1}{4n-2}.$$

They are not equal.

$$-\sum_{n=1}^{10} \frac{(-1)^n}{n!} \frac{1}{4n-1} = .2785055551$$ by calculator.

$$-\sum_{n=1}^{10} \frac{(-1)^n}{n!} \frac{1}{4n-2} = .4406048598$$