Test 3 Fall, 2014 Name: Key

MAT 162 version 1 Gurganus

Directions: Show all work for partial credit purposes. You may use a graphing calculator. The test is closed book.

1. Use Euler's Method to approximate y(1.2) given $\frac{dy}{dx} = 10x - 5y$, and $y(1) = 2$ Use a stepsize of 0.1 .

10

	x	y	f(x,y) h
3	1.	2	$(10-10)(.1) = 0$
3	1.1	2	$(11-10)(.1) = .1$
2	1.2	2.1	

2. Find y(x), the solution to $\frac{dy}{dx} = (y+1)(x^3+x)$, $y(0) = \pi/4$.

18

4 $\int \frac{1}{y+1}\,dy = \int (x^3+x)\,dx$

4 $\ln|y+1| = \frac{1}{4}x^4 + \frac{1}{2}x^2 + C$

2 $\ln(1+\frac{\pi}{4}) = C$

2 $\ln|y+1| = \frac{1}{4}x^4 + \frac{1}{2}x^2 + \ln(1+\frac{\pi}{4})$

2 $|y+1| = e^{\frac{1}{4}x^4 + \frac{1}{2}x^2}(1+\frac{\pi}{4})$

2 $y+1 = (1+\frac{\pi}{4})e^{\frac{1}{4}x^4 + \frac{1}{2}x^2}$

2 $\boxed{y = -1 + (1+\frac{\pi}{4})e^{\frac{1}{4}x^4 + \frac{1}{2}x^2}}$

3. Find y(x), the solution to $\frac{dy}{dx} = 15x + \frac{y}{x}$, $y(1) = 2$.

18

4 $\frac{dy}{dx} - \frac{y}{x} = 15x$

4 $I(x) = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

2 $\frac{d}{dx}(\frac{1}{x}y) = 15$

2 $\frac{1}{x}y = 15x + C$

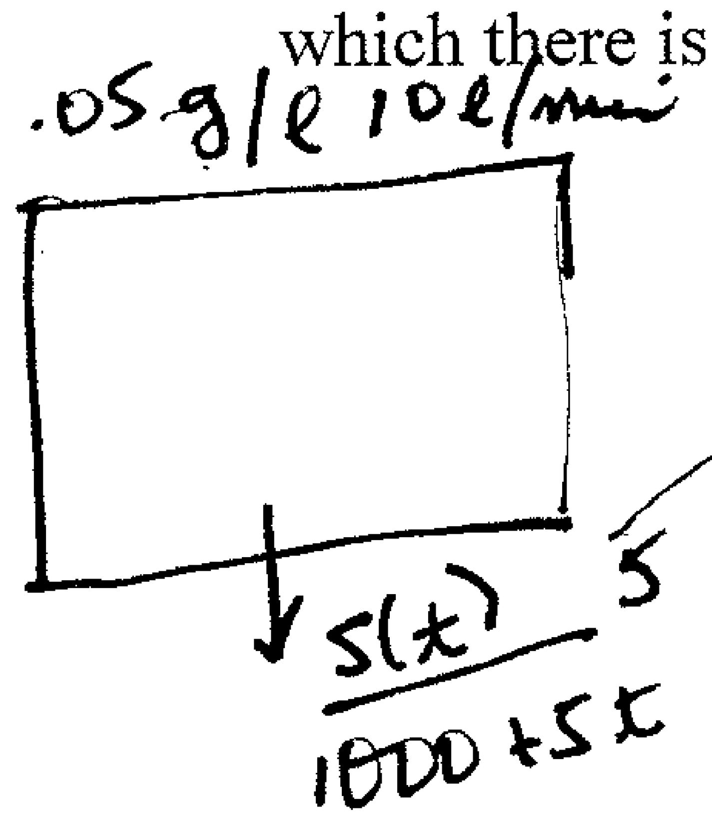
2 $y = 15x^2 + Cx$

2 $2 = y(1) = 15 + C$

$C = -13$

2 $\boxed{y = 15x^2 - 13x}$

4. A tank is filled with 1000 liters of contaminated water containing 5 g of toxins. Water containing .05 g of toxin per liter is pumped in at a rate of 10 l/min., mixes instantaneously, and then is pumped out at a rate 0f 5 l/min.. Find y(t) the number of grams of the toxin in the tank t minutes after the rinse begins. Then find the time at which there is 10 g of toxin present.



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$s(0)=5$

2 $\frac{ds}{dt} = .5 - \frac{1}{200+t} s$

2 $\frac{ds}{dt} + \frac{1}{200+t} s = .5$

2 $I(t) = e^{\int \frac{1}{200+t} dt} = e^{\ln(200+t)} = 200+t$

2 $\frac{d}{dt}((200+t)s) = .5(200+t)$

2 $(200+t)s = .5(200t + \frac{1}{2}t^2) + c$ 2 $\Rightarrow c = 1000$

at $t=0$ $\quad 200(5) = .5(0) + c$

$s = \frac{.5(200t + \frac{1}{2}t^2) + 1000}{200+t}$ 2

(2) $t = 10.7878$ graph

$10 = \frac{.5(200t + \frac{1}{2}t^2) + 1000}{200+t} \Rightarrow 2000 + 10t = 100t + \frac{1}{4}t^2 + 1000$

$\frac{1}{4}t^2 + 90t - 1000 = 0$

$t^2 + 360t - 4000 = 0$

$t = \frac{-360 \pm \sqrt{(360)^2 + 4(4000)}}{2}$

$t + 180 = \pm 10\sqrt{364}$

$t = -180 + 10\sqrt{364} = 10.7878$

$t^2 + 360t + (180)^2 - 4000 - (180)^2$

$(t+180)^2 = 4000 + (180)^2 = 100(40 + 18^2) = 100(364)$

5. First find the solution to $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} - 75y = 0$, $y(0) = 1$, $y'(0) = 2$.

18

2 $r^2 + 10r - 75 = 0$

2 $(r+15)(r-5) = 0$

$r = -15 \quad r = 5$

2 $y(x) = c_1 e^{-15x} + c_2 e^{5x}$

2 $y'(0) = -15c_1 e^{-15x} + 5c_2 e^{5x}$

2 $1 = y(0) = c_1 + c_2$

2 $2 = y'(0) = -15c_1 + 5c_2$

$15 = 15c_1 + 15c_2$

$2 = -15c_1 + 5c_2$

2 $17 = 20c_2 \Rightarrow c_2 = \frac{17}{20}$

$\therefore c_1 = \frac{3}{20}$

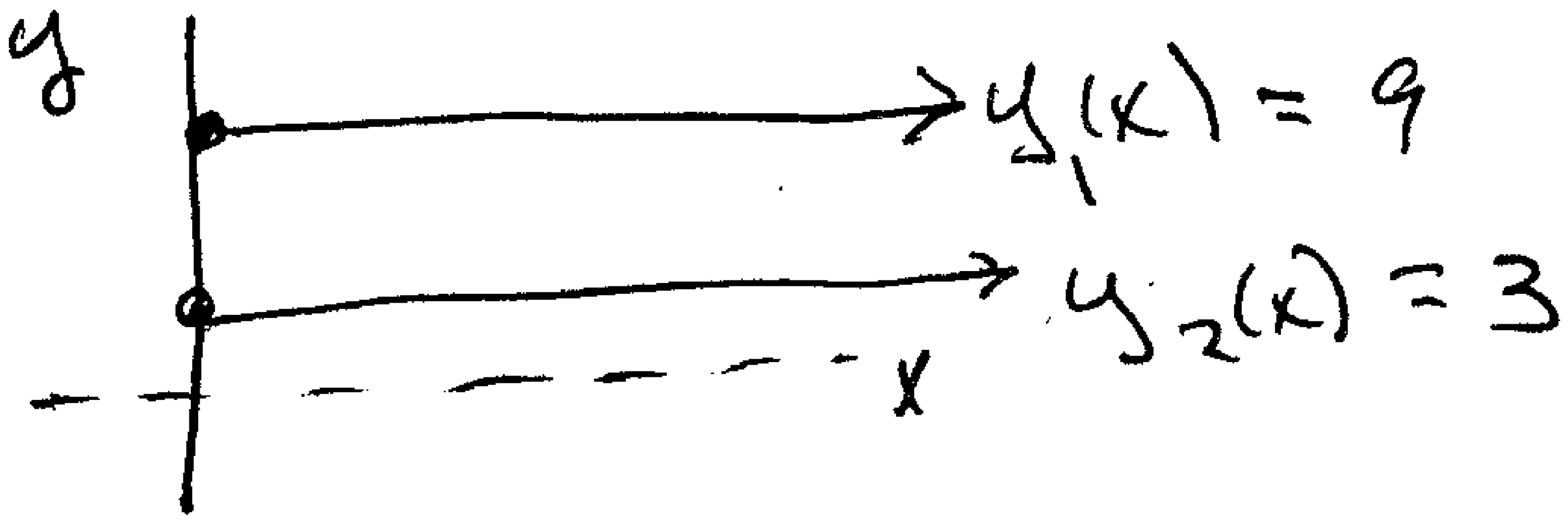
$y(x) = \frac{3}{20}e^{-15x} + \frac{17}{20}e^{5x}$ 2

6. For $\frac{dy}{dx}$ =-2 (y -3)(y-9)

a. Sketch the equilibrium solutions.

18

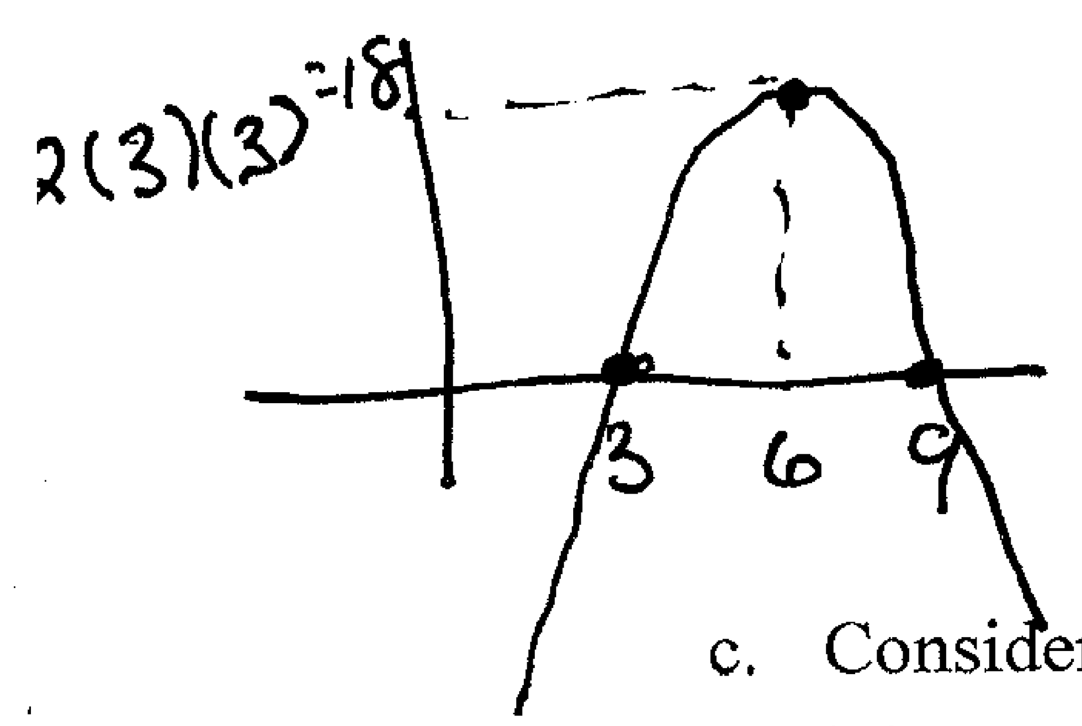
6

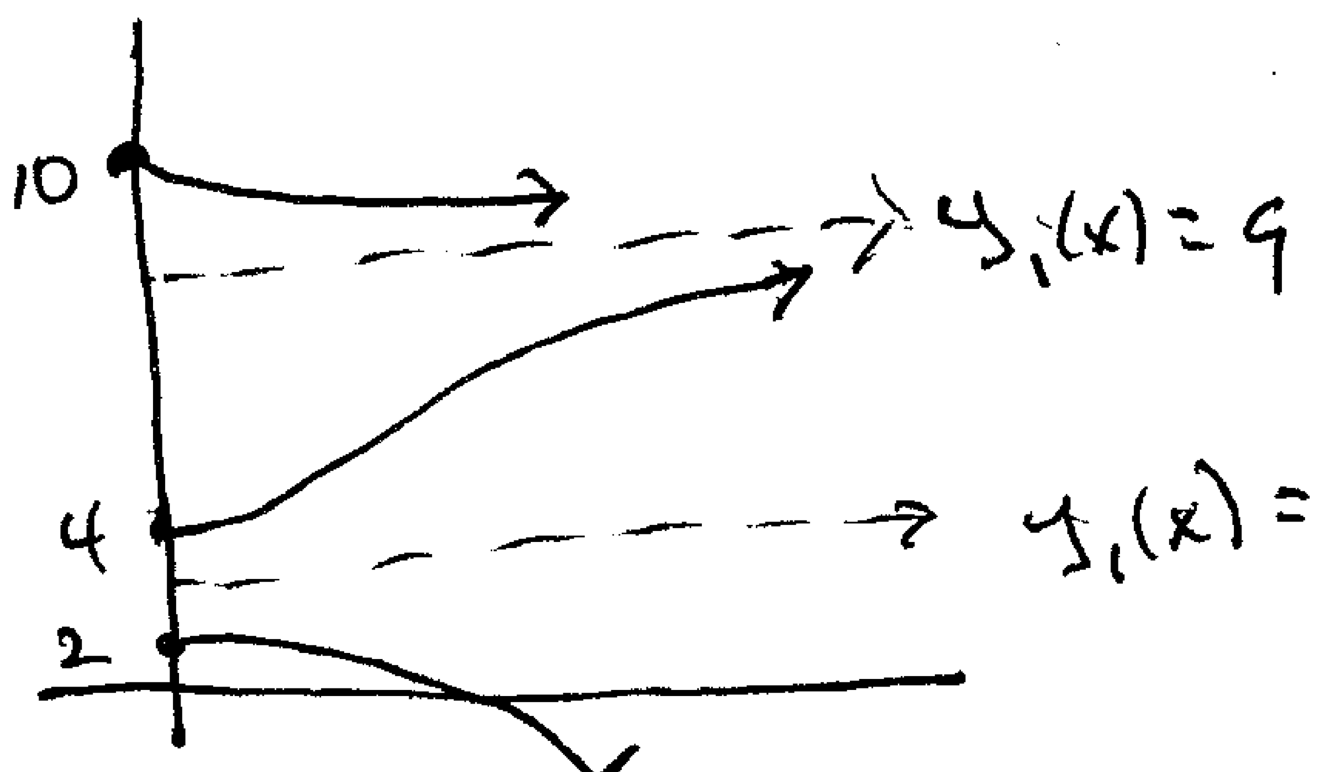


b. Sketch the solutions for the initial conditions y(0) = 2, y(0) = 4, and y(0) = 10.

$z = -2(y-3)(y-9)$

6



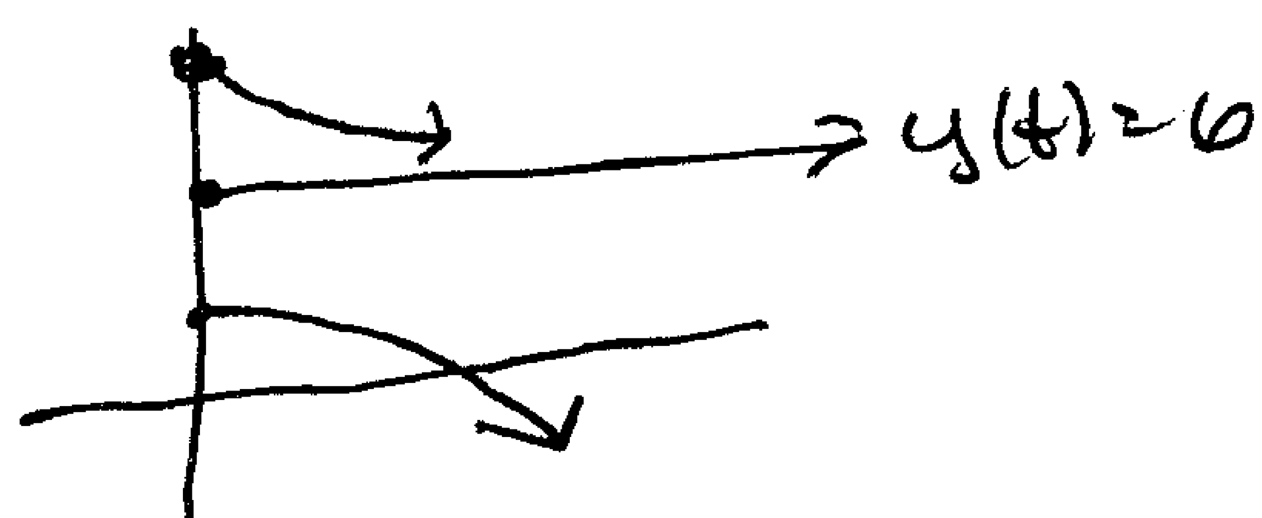


c. Consider $\frac{dy}{dx}$ =-2 (y -3)(y-9) – h where y is a biological population, x is time and h is a constant (harvesting rate). Determine the largest value h can be and not have the population always crash to zero.

6

$h = 18$

$z = -2(y-3)(y-9) - 18$



d. (10 points extra credit) Solve $\frac{dy}{dx}$ =-2 (y -3)(y-9) , y(0) = 4.

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$$\int \frac{1}{(y-3)(y-9)}\,dy = \int -2\,dx \quad 4$$

$$\frac{1}{(y-3)(y-9)} = \frac{A}{y-3} + \frac{B}{y-9} = \frac{1}{6}\left[\frac{1}{y-9} - \frac{1}{y-3}\right]$$

$1 = A(y-9) + B(y-3)$

$1 = A(3-9) + B(0) \Rightarrow A = -\frac{1}{6}$

$1 = A(0) + B(6) \quad = \quad B = \frac{1}{6}$

4

$$\frac{1}{6}\left[\ln|y-9| - \ln|y-3|\right] = -2x + C$$

$$\frac{1}{6}\ln\left|\frac{y-9}{y-3}\right| = -2x + C$$

$$\ln\left|\frac{y-9}{y-3}\right| = -12x + K$$

$$\frac{y-9}{y-3} = \hat{K}e^{-12x}$$

$$\frac{-5}{1} = \frac{4-9}{4-3} = \hat{K}$$

$$\frac{y-9}{y-3} = -\frac{5}{1}e^{-12x} = w$$

$y - 9 = wy - 3w \Rightarrow y(1-w) = 9 - 3w$

$$y = \frac{9-3w}{\ldots} = \frac{9 - 3(-\frac{5}{1}e^{-12x})}{\ldots} \quad \frac{9+15e}{1+5e}$$

Test 3 Fall, 2014 Name: Key

MAT 162 version 2 Gurganus

Directions: Show all work for partial credit purposes. You may use a graphing calculator. The test is closed book.

1. Use Euler's Method to approximate y(2.2) given $\frac{dy}{dx} = 5x - 10y$, and $y(2) = 1$ Use a stepsize of 0.1 .

10

	x	y	f(x,y)h
3	2	1	$(0)(.1) = 0$
3	2.1	1	$(10.5 - 10)(.1) = .05$
2	2.2	1.05	

2. Find y(x), the solution to $\frac{dy}{dx} = (2y + 1)(x^2 + 7x)$, $y(0) = \pi/4$.

18

4 $\int \frac{1}{2y+1}\,dy = \int (x^2+7x)\,dx$

3 $\frac{1}{2}\ln|2y+1| = \frac{1}{3}x^3 + \frac{7}{2}x^2 + C$

$2\ln|2y+1| = \frac{2}{3}x^3 + 7x^2 + C$

2 $2y+1 = Ke^{2/3x^3+7x^2}$

2 $2(\frac{\pi}{4}) + 1 = K = \frac{\pi}{2} + 1$

$2y+1 = (\frac{\pi}{2}+1)e^{2/3x^3+7x^2}$

2 $\boxed{y = \dfrac{-1 + (\frac{\pi}{2}+1)e^{2/3x^3+7x^2}}{2}}$

3. Find y(x), the solution to $\frac{dy}{dx} = 15x + \frac{y}{2x}$, $y(1) = 3$.

18

$\frac{dy}{dx} - \frac{y}{2x} = 15x$

4 $I(x) = e^{-\int \frac{1}{2x}dx} = e^{-\frac{1}{2}\ln x} = x^{-\frac{1}{2}}$

2 $\frac{d}{dx}(x^{-\frac{1}{2}}y) = 15x^{\frac{1}{2}}$

2 $x^{-\frac{1}{2}}y = 15(\frac{2}{3})x^{3/2} + C$

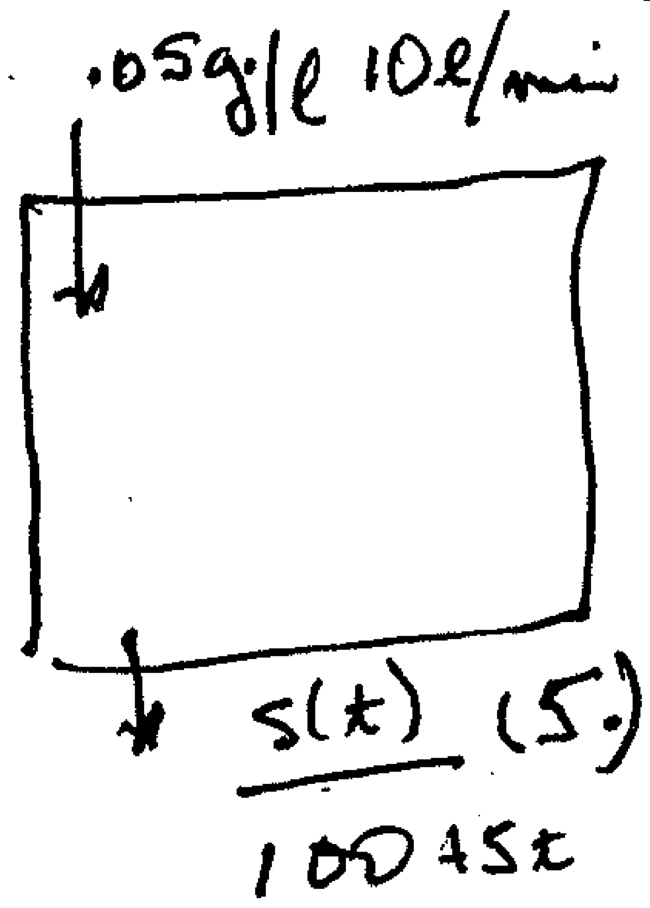
2 $x^{-\frac{1}{2}}y = 10x^{3/2} + C$

$3 = 10 + C$

2 $-7 = C$

$\boxed{y = 10x^2 - 7x^{\frac{1}{2}}}$

4. A tank is filled with 100 liters of contaminated water containing 5 g of toxins. Water containing .05 g of toxin per liter is pumped in at a rate of 10 l/min., mixes instantaneously, and then is pumped out at a rate 0f 5 l/min.. Find y(t) the number of grams of the toxin in the tank t minutes after the rinse begins. Then find the time at which there is 10 g of toxin present.



18

2 $S(0) = 5$

2 $\frac{ds}{dt} = .5 - \frac{1}{20+t} S$

2 $\frac{ds}{dt} + \frac{1}{20+t} S = .5$

2 $I(t) = e^{\int \frac{1}{20+t} d} = 20+t$

2 $\frac{d}{dt}\left((20+t)(y)\right) = .5(20+t) = 10 + \frac{1}{2}t$

2 $(20+t)y = 10t + \frac{1}{4}t^2 + C$

2 $(20)(5) = 0 + C \Rightarrow C = 100$

2 $y = \frac{10t + \frac{1}{4}t^2 + 100}{20+t}$

2

$10 = \frac{10t + \frac{1}{4}t^2 + 100}{20+t}$

$\frac{1}{4}t^2 + \cancel{10t} + 100 = 200 + \cancel{10t}$

$\frac{1}{4}t^2 = 100$

$t^2 = 400$

$\boxed{t = 20 \text{ seconds}}$

5. First find the solution to $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} - 75y = 0$, $y(0) = 1$, $y'(0) = 2$.

18

2 $r^2 - 10r - 75 = 0$

2 $(r-15)(r+5) = 0$

2 $r = 15,\ r = -5$

2 $y(x) = c_1 e^{15x} + c_2 e^{-5x}$

2 $y'(x) = 15c_1 e^{15x} - 5c_2 e^{-5x}$

2 $\begin{cases} 1 = y(0) = c_1 + c_2 \\ 2 = 15c_1 - 5c_2 \end{cases}$

$5 = 5c_1 + 5c_2$

$2 = 15c_1 - 5c_2$

$7 = 20c_1$ $\quad c_1 = \frac{7}{20}$

2

2 $c_2 = \frac{13}{20}$

$y = \frac{7}{20} e^{15x} + \frac{13}{20} e^{-5x}$

6. For $\frac{dy}{dx} = -2\,(y-3)(y-7)$

a. Sketch the equilibrium solutions.

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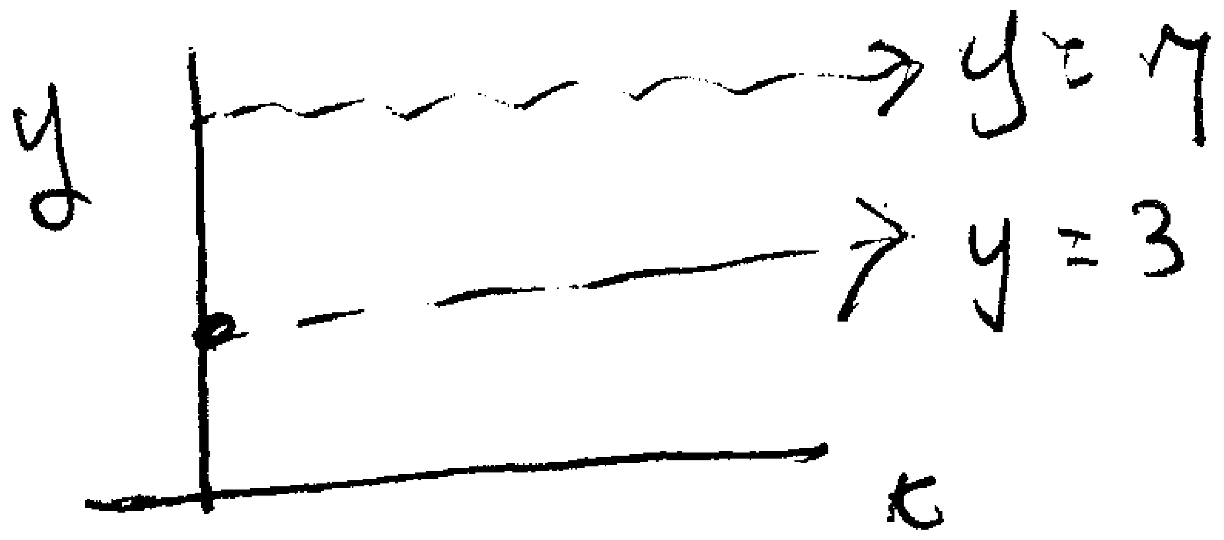
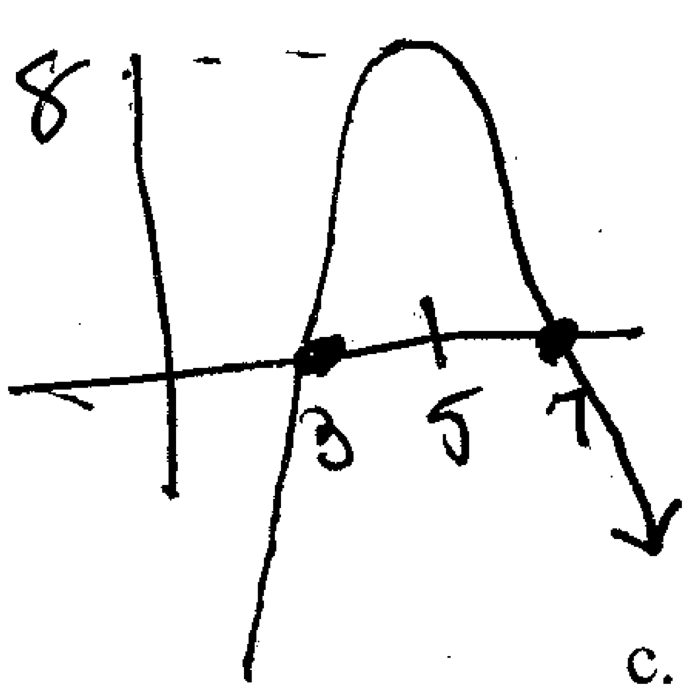


b. Sketch the solutions for the initial conditions y(0) = 2, y(0) = 4, and y(0) = 10.

$z = -2(y-3)(y-7)$



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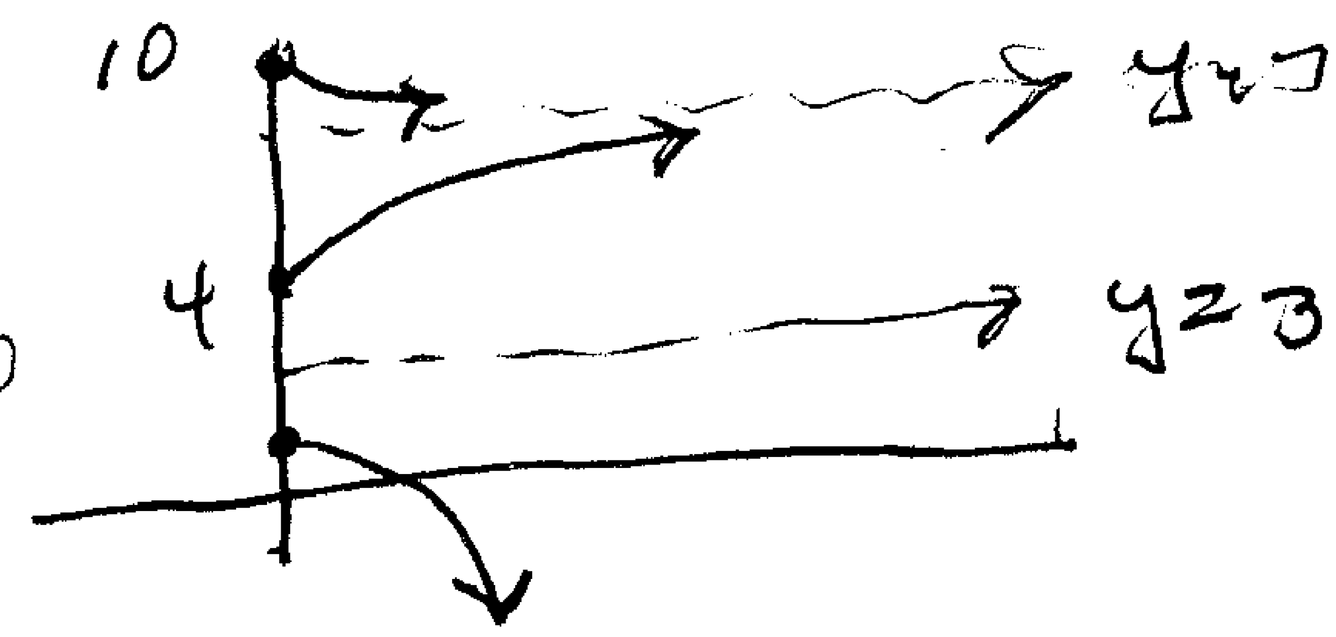
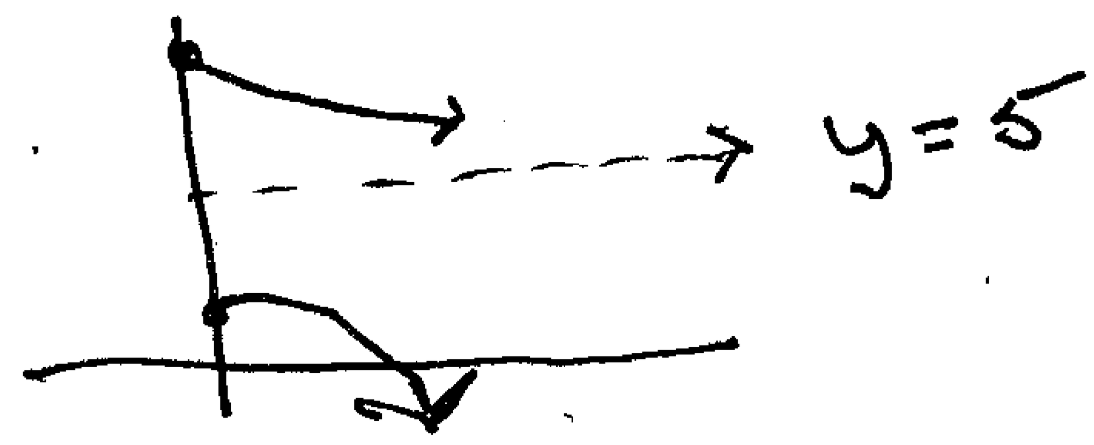


c. Consider $\frac{dy}{dx} = -2\,(y-3)(y-7) - h$ where y is a biological population, x is time and h is a constant (harvesting rate). Determine the largest value h can be and not have the population always crash to zero.

6

h = 8



d. (10 points extra credit) Solve $\frac{dy}{dx} = -2\,(y-3)(y-7)$, y(0) = 4.

$\int \frac{1}{(y-3)(y-7)}\,dy = \int -2\,dx$ 4

$\frac{1}{(y-3)(y-7)} = \frac{A}{y-3} + \frac{B}{y-7} = \frac{1}{4}\left[\frac{1}{y-7} - \frac{1}{y-3}\right]$

$1 = A(y-7) + B(y-3)$

$1 = A(3-7) + B(0) \Rightarrow A = -\frac{1}{4}$

$1 = A(0) + B(4) \Rightarrow B = \frac{1}{4}$

$\frac{1}{4}\left[\ln|y-7| - \ln|y-3|\right] = -2x + c$

$\frac{1}{4}\ln\left|\frac{y-7}{y-3}\right| = -2x + c$ ✓

$\ln\left|\frac{y-7}{y-3}\right| = -8x + K$

$\frac{y-7}{y-3} = \hat{K}e^{-8x}$

$-3 = \frac{4-7}{4-3} = \hat{K}$

$\frac{y-7}{y-3} = -3e^{-8x} = w$

$y-7 = w(y-3)$

$y = \frac{7-3w}{1-w} = \frac{7+9e^{-8x}}{1+3e^{-8x}}$