Fall, 2018 Test 1 Name: Key

MAT 162001 Gurganus

Directions: Show all work for partial credit purposes. You may use a graphing calculator and any notes you may put on one side of an 8.5 by 11 inch sheet of paper. Otherwise the test is closed book. When you turn in your test, staple your notes to this sheet.

For 1-4, calculate the following:

1. $\int x^2 \ln(x)\,dx$ $= uv - \int v\,du = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \frac{1}{x}\,dx$

$u = \ln x \quad dv = x^2\,dx$

$du = \frac{1}{x}\,dx \quad v = \frac{1}{3}x^3$

$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2\,dx$

$= \boxed{\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C}$

2. $\int cos^5(x) sin^2(x)dx$ $= \int \cos^4 x \cdot \sin^2 x \cos x\,dx$

$= \int (1-\sin^2 x)^2 \sin^2 x \cos x\,dx$

$u = \sin x$

$du = \cos x\,dx$

$= \int (1-u^2)^2 u^2\,du = \int (u^2 - 2u^4 + u^6)\,du$

$= \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$

$= \boxed{\frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C}$

3. $\int x^3 \sqrt{36 - x^2}\,dx$ $= \int 6^3 \sin^3\theta\, 6^2 \cos^2\theta\,d\theta$

$x = 6\sin\theta$

$\sqrt{36-x^2} = 6\cos\theta$

$dx = 6\cos\theta\,d\theta$

$= 6^5 \int (1-\cos^2\theta)\cos^2\theta \sin\theta\,d\theta$

$= 6^5 \int (1-w^2)w^2(-dw)$

$w = \cos\theta = \frac{\sqrt{36-x^2}}{6}$

$dw = -\sin\theta\,d\theta$

$-dw = \sin\theta\,d\theta$

$= -6^5 \int (w^2 - w^4)\,dw$

$= -6^5 \left(\frac{1}{3}w^3 - \frac{1}{5}w^5\right) + C$

$= -6^5\left[\frac{1}{3}\left(\frac{\sqrt{36-x^2}}{6}\right)^3 - \frac{1}{5}\left(\frac{\sqrt{36-x^2}}{6}\right)^5\right] + C$

4. $\int \frac{x+4}{x^2-16x+60}\,dx$

$$\frac{x+4}{(x-10)(x-6)} = \frac{A}{x-10} + \frac{B}{x-6} = \frac{7}{2}\frac{1}{x-10} - \frac{5}{2}\frac{1}{x-6}$$

3

$x+4 = A(x-6) + B(x-10)$ $\quad \therefore \int \frac{x+4}{x^2-16x+60}\,dx = \frac{7}{2}\ln|x-10| - \frac{5}{2}\ln|x-6| + C$

$x=10 \quad 14 = A(4) \Rightarrow A = \frac{7}{2}$

$x=6 \quad 10 = B(-4) = \quad B = -\frac{5}{2}$

5. Estimate $\int_4^6 e^{1+\sin(x)}\,dx$ using the ~~Simpson~~ Midpoint's Rule with n = ~~6~~ 4. Write the sum; you do not have to evaluate the sum.

4.5 5.5

4 5 6

$\Delta x = \frac{6-4}{4} = \frac{1}{2}$ $\qquad f(x) = e^{1+\sin x}$

$\Delta x\left(f(4.25) + f(4.75) + f(5.25) + f(5.75)\right)$

2

6. Calculate the following; it the integral does not converge, state "does not converge."

a. $\int_2^{+\infty} \frac{1}{x^2+6x+5}\,dx \quad = \lim_{t\to+\infty}\int_2^t \frac{1}{x^2+6x+5}\,dx$

$$\frac{1}{(x+5)(x+1)} = \frac{A}{x+1} + \frac{B}{x+5}$$

$1 = A(x+5) + B(x+1)$

$x=-1 \quad 1 = A(4) \Rightarrow A = \frac{1}{4}$

$x=-5 \quad 1 = B(-4) \Rightarrow B = -\frac{1}{4}$

$\int \frac{1}{(x+5)(x+1)}\,dx = \frac{1}{4}\ln|x+1| - \frac{1}{4}\ln|x+5| + C$

$= \frac{1}{4}\ln\left|\frac{x+1}{x+5}\right|$

$= \lim_{t\to+\infty} \frac{1}{4}\ln\left|\frac{x+1}{x+5}\right|\Big|_2^t$

$= -\frac{1}{4}\ln\left|\frac{2+1}{2+5}\right| = \boxed{\frac{1}{4}\ln\frac{7}{3}}$

b. $\int_4^5 \frac{2x}{\sqrt{x^2-16}}\,dx \quad = \lim_{t\to4^+}\int_t^5 \frac{2x}{\sqrt{x^2-16}}\,dx$

8

$= \lim_{t\to4^+} 2(x^2-16)^{\frac{1}{2}}\Big|_t^5$

$= \lim_{t\to4^+} 2(25-16)^{\frac{1}{2}} - 2(t^2-16)^{\frac{1}{2}}$

$= 2(9^{\frac{1}{2}}) = \boxed{6}$ 2

7. Tell why the following converge or diverge:

a. $\int_4^{+\infty} \frac{2x}{\sqrt{x^4-16}}\, dx$

$\frac{2}{x} = \frac{2x}{\sqrt{x^4}} \leq \frac{2x}{\sqrt{x^4-16}}$

$\int_4^{+\infty} \frac{2}{x}\, dx$ diverges ($p=1$)

$\therefore \int_4^{+\infty} \frac{2x}{\sqrt{x^2-16}}\, dx$ diverges

b. $\int_4^{+\infty} \frac{2x}{\sqrt{x^6+16}}\, dx$

$\frac{2x}{\sqrt{x^6+16}} \leq \frac{2x}{\sqrt{x^6}} \leq \frac{2x}{x^3} = \frac{2}{x^2}$

$\int_4^{+\infty} \frac{2}{x^2}\, dx$ converges

$\therefore \int_4^{+\infty} \frac{2x}{\sqrt{x^6+16}}\, dx$ converges ~~diverges~~

8. Calculate $\int \frac{x+4}{x^2-16x+68} dx$.

$\frac{x+4}{x^2-16x+64+4} = \frac{x+4}{(x-8)^2+2^2}$

let $u = x-8$
$u+12 = x+4$
$du = dx$

$= \int \frac{u+12}{u^2+2^2}\, du$

$= \frac{1}{2}\int \frac{2u}{u^2+2^2}\, du + 12\int \frac{1}{u^2+2^2}\, du$

$= \frac{1}{2}\ln|u^2+2^2| + 12\cdot\frac{1}{2}\arctan\frac{u}{2} + C$

$= \frac{1}{2}\ln|x^2-16x+68| + 6\arctan\frac{x-8}{2} + C$

9. Write the partial fraction decomposition form for the following rational function. You do not have to solve for the constants.

$$\frac{(x^3+5)}{(x^2+11)^3(x^2+12x)^2}$$

$\frac{A_1x+B_1}{x^2+11} + \frac{A_2x+B_2}{(x^2+11)^2} + \frac{A_3x+B_3}{(x^2+11)^3}$

$+ \frac{C}{x} + \frac{D}{x^2} + \frac{E}{x+12} + \frac{F}{(x+12)^2}$

Fall, 2018 Test1 Name: Key

MAT 162002 Gurganus

Directions: Show all work for partial credit purposes. You may use a graphing calculator and any notes you may put on one side of an 8.5 by 11 inch sheet of paper. Otherwise the test is closed book. When you turn in your test, staple your notes to this sheet.

For 1-4, calculate the following:

1. $\int x^5 \ln(x)\, dx$ $= uv - \int v\, du$

$u = \ln x \quad dv = x^5 dx$

$du = \frac{1}{x} dx \quad v = \frac{1}{6}x^6$

$= \frac{1}{6}x^6 \ln x - \int \frac{1}{6}x^6 \frac{1}{x} dx$

$= \frac{1}{6}x^6 \ln x - \frac{1}{6}\int x^5 dx$

$= \frac{1}{6}x^6 \ln x - \frac{1}{36}x^6 + C$

2. $\int cos^7(x) sin^4(x)dx$

$\int \cos^6(x)\sin^4 x \cos x\, dx = \int (1-\sin^2 x)^3 \sin^4 x \cos x\, dx$

$u = \sin x$

$du = \cos x\, dx$

$= \int (1-u^2)^3 u^4\, du$

$= \int (1 - 3u^2 + 3u^4 - u^6) u^4\, du$

$= \int (u^4 - 3u^6 + 3u^8 - u^{10})\, du = \frac{1}{5}u^5 - \frac{3}{7}u^7 + \frac{3}{9}u^9 - \frac{1}{11}u^{11} + C$

$= \boxed{\frac{1}{5}\sin^5 x - \frac{3}{7}\sin^7 x + \frac{3}{9}\sin^9 x - \frac{1}{11}\sin^{11} x + C}$

3. $\int x^3 \sqrt{x^2 - 36} dx$

$x = 6\sec\theta$

$\sqrt{x^2-36} = 6\tan\theta$

$dx = 6\sec\theta\tan\theta\, d\theta$

$= \int 6^3\sec^3\theta \cdot 6\tan\theta \; 6\sec\theta\tan\theta\, d\theta$

$= 6^5 \int \sec^4\theta \tan^2\theta\, d\theta$

$= 6^5 \int \sec^2\theta \tan^2\theta \sec^2\theta\, d\theta$

$= 6^5 \int (\tan^2\theta + 1)\tan^2\theta \sec^2\theta\, d\theta$

$u = \tan\theta = \frac{\sqrt{x^2-36}}{6}$

$du = \sec^2\theta\, d\theta$

$= 6^5 \int (u^2+1)u^2\, du = 6^5 \int (u^4+u^2)\, du$

$= 6^5 \left[\frac{1}{5}u^5 + \frac{1}{3}u^3\right] + C$

$= 6^5 \left[\frac{1}{5}\left[\frac{\sqrt{x^2-36}}{6}\right]^5 + \frac{1}{3}\left[\frac{\sqrt{x^2-36}}{6}\right]^3\right] + C$

10 4. $\int \frac{x+4}{x^2-32x+60}\,dx$ $= \int -\frac{3}{14}\frac{1}{x-2} + \frac{17}{14}\frac{1}{x-30}\,dx$

$$\frac{x+4}{(x-2)(x-30)} = \frac{A}{x-2} + \frac{B}{x-30}$$ 3

$= -\frac{3}{14}\ln|x-2| + \frac{17}{14}\ln|x-30| + C$

2 2 1

$x+4 = A(x-30) + B(x-2)$

$x=2$ $6 = A(-28) \Rightarrow A = -\frac{3}{14}$ 1

$x=30$ $34 = B(28)$ 1

$\therefore B = \frac{17}{14}$

5. Estimate $\int_4^6 e^{1+\sin(x)}\,dx$ using the Simpson's Rule with n = 4. Write the sum; you do not have to evaluate the sum.

(8) $f(x) = e^{1+\sin x}$

4 4.5 5 5.5 6 1

$\Delta x = \frac{6-4}{4} = \frac{1}{2}$ 1

$$\frac{\frac{1}{2}}{3}\left[f(4) + 4f(4.5) + 2f(5) + 4f(5.5) + f(6)\right]$$

1 $\frac{1}{2}$ 1 1 1 $\frac{1}{2}$

6. Calculate the following; it the integral does not converge, state "does not converge."

8 a. $\int_2^{+\infty} \frac{1}{x^2+8x+7}\,dx$ $= \lim_{t\to+\infty} \int_2^t \frac{1}{x^2+8x+7}\,dx$

$\frac{1}{x^2+8x+7} = \frac{1}{(x+7)(x+1)}$ 2

$= \frac{A}{x+7} + \frac{B}{x+1}$

$1 = A(x+1) + B(x+7)$

$x=-7$ $1 = A(-6) \Rightarrow A = -\frac{1}{6}$ 2

$x=-1$ $1 = B(6) \Rightarrow B = \frac{1}{6}$

$= \lim_{t\to+\infty} \left[-\frac{1}{6}\ln|x+7| + \frac{1}{6}\ln|x+1|\right]_2^t$ 2

$= \lim_{t\to+\infty} \frac{1}{6}\ln\left|\frac{t+1}{t+7}\right| - \frac{1}{6}\ln\left|\frac{3}{9}\right|$ 1

1

$= \frac{1}{6}\ln\frac{9}{3} = \boxed{\frac{1}{6}\ln 3}$ 1

8 b. $\int_6^8 \frac{2x}{\sqrt{x^2-36}}\,dx$ $= \lim_{t\to6^+} \int_t^8 \frac{2x}{\sqrt{x^2-36}}\,dx$

$= \lim_{t\to6^+} 2\sqrt{x^2-36}\,\Big|_t^8$

$= 2\sqrt{64-36} = 2\sqrt{28} = 4\sqrt{7}$

7. Tell why the following converge or diverge:

(8)

a. $\int_4^{+\infty} \frac{2x}{\sqrt{x^4-16x}}\,dx$

$\frac{2}{x} = \frac{2x}{x^2} = \frac{2x}{\sqrt{x^4}} \le \frac{2x}{\sqrt{x^4-16x}}$ 4

$\int_4^{+\infty} \frac{2}{x}\,dx$ diverges 2

$\therefore \int_4^{+\infty} \frac{2x}{\sqrt{x^4-16x}}\,dx$ diverges 2

(8)

b. $\int_4^{+\infty} \frac{2x}{\sqrt{x^6+32x+7}}\,dx$

4 $\frac{2x}{\sqrt{x^6+32x+7}} \le \frac{2x}{\sqrt{x^6}} = \frac{2x}{x^3} = \frac{2}{x^2}$

2 $\int_4^{+\infty} \frac{2}{x^2}\,dx$ converges

2 $\therefore \int_4^{+\infty} \frac{2x}{\sqrt{x^6+32x+7}}\,dx$ ~~diverges~~ converges

(10) 8. Calculate $\int \frac{x+4}{x^2-20x+101}\,dx$

$\int \frac{x+4}{x^2-20x+101}\,dx = \int \frac{x+4}{(x-10)^2+1}\,dx = \int \frac{u+10+4}{u^2+1}\,du$ 2 1

$u = x-10$
$u+10 = x$
$du = dx$

$= \frac{1}{2}\int \frac{2u}{u^2+1}\,du + 14\int \frac{1}{u^2+1}\,du$ 1 1

$= \frac{1}{2}\ln|u^2+1| + 14\arctan u + C$ 1

$= \boxed{\frac{1}{2}\ln|x^2-20x+101| + 14\arctan(x-10) + C}$ 1 1 1

(6) 9. Write the partial fraction decomposition form for the following rational function. You do not have to solve for the constants.

$$\frac{(x^3+5)}{(x^2+11)^2(x^2+12x)^3}$$

$\frac{A_1x+B_1}{x^2+11} + \frac{A_2x+B_2}{(x^2+11)^2} + \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{C_3}{x^3} + \frac{D_1}{x+12} + \frac{D_2}{(x+12)^2} + \frac{D_3}{(x+12)^3}$

2 2 1 1 1 1 1 1

Fall, 2018
MAT 162001

Name: Key

Test 2 Version 1

Gurganus

Directions: Show all work for partial credit purposes. You may use a graphing calculator. Otherwise the test is closed book. Each problem except 2 is worth 14 points. Problem 2 is 16 points.

1. Find the length of the curve $y = (2x+7)^{1.5}$, $3 \le x \le 5$.

14

$$L = \int_3^5 \sqrt{1+(f'(x))^2}\,dx$$

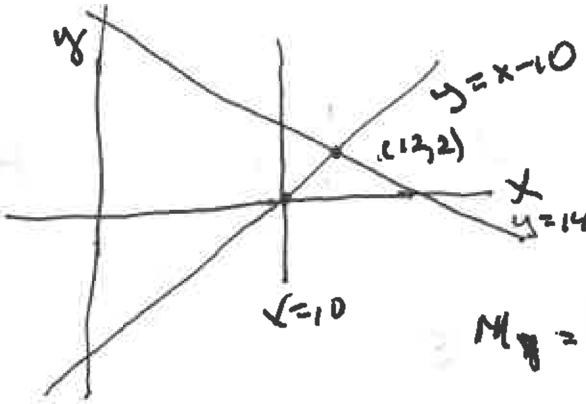
$$= \int_3^5 \sqrt{1+\left[(1.5)(2)(2x+7)^{.5}\right]^2}\,dx$$

$$= \int_3^5 \sqrt{1+9(2x+7)}\,dx = \int_3^5 \sqrt{64+18x}\,dx$$

$$= (64+18x)^{3/2}\left(\frac{2}{3}\right)\frac{1}{18}\Big|_3^5 = \frac{1}{27}\left[154^{3/2} - 118^{3/2}\right] = 23.307$$

2. Find the centroid (center of mass) of the region bounded by the curves: $y = x - 10$ and $y = 14 - x$, and $x = 10$.

16



$$\text{Mass} = \int_{10}^{12} \rho\,(14-x-(x-10))\,dx$$

$$= \int_{10}^{12} \rho\,(24-2x)\,dx = \rho(24x - x^2)\Big|_{10}^{12}$$

$$= \rho(144-140) = 4\rho$$

$$M_y = \int_{10}^{12} \rho\,(24x - 2x^2)\,dx = \rho\left(12x^2 - \frac{2}{3}x^3\right)\Big|_{10}^{12}$$

$$= \rho\left(576 - 533\tfrac{1}{3}\right) = \rho\left(42\tfrac{2}{3}\right)$$

$$M_x = \int_{10}^{12} \frac{\rho}{2}\left((14-x)^2 - (x-10)^2\right)dx$$

$$= \int_{10}^{12} \frac{\rho}{2}\left(14^2 - 28x - (-20x+100)\right)dx$$

$$= \int_{10}^{12} \frac{\rho}{2}(96 - 8x)\,dx = \int_{10}^{12} \rho(48 - 4x)\,dx$$

$$\rho(48x - 2x^2)\Big|_{10}^{12} = \rho(288 - 280) = 8\rho$$

$y = x - 10$
$y = -x + 14$
$2y = 4$
$y = 2$
$x = 12$

$$\bar{x} = \frac{M_y}{\text{mass}} = \frac{42\tfrac{2}{3}}{4} = 10\tfrac{2}{3}$$

$$\bar{y} = \frac{M_x}{\text{mass}} = \frac{8\rho}{4\rho} = 2$$

$(10\tfrac{2}{3}, 2)$ is the centroid

3. Find y(x), the solution to $y^2 \frac{dy}{dx} = (x^2 + 1)$, y(0) = 1.

(14)

$\int y^2 dy = \int (x^2+1) dx$

$\frac{1}{3}y^3 = \frac{1}{3}x^3 + x + C$

$\frac{1}{3} = 0 + C$

$\frac{1}{3}y^3 = \frac{1}{3}x^3 + x + \frac{1}{3}$

$y^3 = x^3 + 3x + 1$

$y = (x^3+3x+1)^{\frac{1}{3}}$

4. Let f(x) = (x^{-6} + Ax^{-8}) for x $\geq$ 1 (for x < 1, f(x) = 0).

a. Find the value of A in order that f(x) is a probability density function.

(14)

$1 = \int_1^{+\infty} (x^{-6} + Ax^{-8}) dx = \lim_{b \to +\infty} \left(\frac{1}{5}x^{-5} - \frac{1}{7}Ax^{-7}\right)\Big|_1^b = \frac{1}{5} + \frac{A}{7}$

(7)

$\frac{4}{5} = \frac{A}{7}$

$A = \frac{28}{5}$

b. Find the mean of the probability density function.

(7)

$\mu = \int_1^{+\infty} (x^{-5} + \frac{28}{5}x^{-7}) dx = \lim_{b \to +\infty} -\frac{1}{4}x^{-4} + \frac{28}{5}(-\frac{1}{6}x^{-6})\Big|_1^b$

$= \frac{1}{4} + \frac{28}{5}(\frac{1}{6}) = \frac{1}{4} + \frac{28}{30} = \frac{15}{60} + \frac{56}{60} = \frac{71}{60}.$

5. Find y(x), the solution to $\frac{dy}{dx} + \frac{\cos(x)y}{\sin(x)} = 1 \cdot y(\pi/2) = 2$.

$I(x) = e^{\int \frac{\cos x}{\sin x} dx} = e^{\ln \sin x} = \sin x$ 4

14

$\frac{d}{dx}(y \sin x) = \sin x$ 4

$y \sin x = -\cos x + C$

$y = \frac{-\cos x + C}{\sin x}$ 4

$2 = y(\frac{\pi}{2}) = C$ 1

$y = \frac{-\cos x + 2}{\sin x}$ 1

6. Find the solution to $\frac{d^2 y}{dx^2} + 15\frac{dy}{dx} - 16y = 0$, y(0) = 1, y'(0) = 4.

14

$r^2 + 15r - 16 = 0$

$(r + 16)(r - 1) = 0$

$r = -16 \quad r = 1$

$y(x) = c_1 e^{-16x} + c_2 e^{x}$; $y'(x) = -16 c_1 e^{-16x} + c_2 e^{x}$

$1 = c_1 + c_2$

$4 = -16 c_1 + c_2$

$-3 = 17 c_1 + 0$

$c_1 = -\frac{3}{17}$

$c_2 = 1 - c_1 = 1 - (-\frac{3}{17}) = \frac{20}{17}$

$y(x) = -\frac{3}{17} e^{-16x} + \frac{20}{17} e^{x}$

$\boxed{y(x) = -\frac{3}{17} e^{-16x} + \frac{20}{17} e^{x}}$

7. A biological population is growing at a rate directly proportional to the size of the population. At t = 0 hours, the population is 3 units and at t= 4 hours the population is 5 units. Find the population at t = 7 hours.

14

$P(t) = P_0 e^{kt}$

$3 = P(0) \Rightarrow P_0 = 3$

$P(t) = 3e^{kt}$

$5 = P(4) = 3e^{k4}$

$\frac{5}{3} = e^{k4}$

$\ln \frac{5}{3} = k4$

$k = \frac{1}{4} \ln \frac{5}{3}$

$P(7) = 3e^{\frac{7}{4} \ln \frac{5}{3}}$

$= 3\left(\frac{5}{3}\right)^{7/4}$

$= 7.334$

Fall, 2018 Name: Key

MAT 16200 Test 2 Version 2 Gurganus

Directions: Show all work for partial credit purposes. You may use a graphing calculator. Otherwise the test is closed book. Each problem except 2 is worth 14 points. Problem 2 is 16 points.

1. Find the length of the curve $y = (x+7)^{1.5}$, $3 \le x \le 5$.

14

$$L = \int_3^5 \sqrt{1+(f'(x))^2}\,dx$$

$$= \int_3^5 \sqrt{1+\left(1.5(x+7)^{.5}\right)^2}\,dx$$

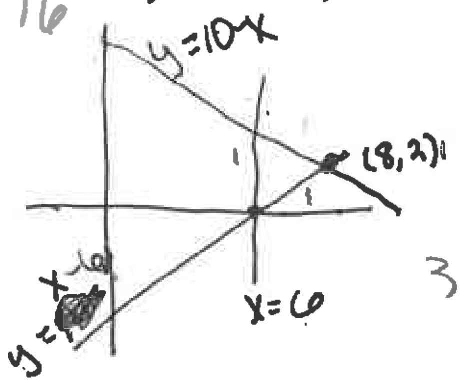
$$= \int_3^5 \sqrt{1+2.25(x+7)}\,dx = \int_3^5 \sqrt{16.75+2.25x}\,dx$$

$$= \frac{1}{2.25}\cdot\frac{2}{3}(16.75+2.25x)^{3/2}\Big|_3^5 = \frac{2}{6.75}(16.75+2.25x)^{3/2}\Big|_3^5$$

$$= \frac{2}{6.75}\left(28^{3/2} - 23.5^{3/2}\right) = 10.1456583$$

2. Find the centroid (center of mass) of the region bounded by the curves: $y = x - 6$ and $y = 10 - x$, and $x = 6$.

16



$$\text{Mass} = \int_6^8 \rho\left((10-x)-(x-6)\right)dx$$

$$= \int_6^8 \rho(16-2x)\,dx = \rho(16x-x^2)\Big|_6^8$$

$$= \rho\left(16\cdot 8 - 8^2 - (16\cdot 6 - 6^2)\right) = 4\rho$$

$$M_y = \int_6^8 \rho(16x-2x^2)\,dx = \rho\left(8x^2 - \tfrac{2}{3}x^3\right)\Big|_6^8$$

$$= \rho\left(170\tfrac{2}{3} - 144\right) = \rho\left(26\tfrac{2}{3}\right)$$

$$M_x = \int_6^8 \frac{\rho}{2}\left((10-x)^2-(x-6)^2\right)dx$$

$$= \int_6^8 \frac{\rho}{2}(64-20x+12x) = \frac{\rho}{2}\left(64x - \tfrac{8}{2}x^2\right)\Big|_6^8$$

$$= \rho(32x-2x^2)\Big|_6^8 = \rho(8)$$

$$\bar{x} = \frac{M_y}{\text{Mass}} = \frac{\rho\left(26\frac{2}{3}\right)}{4\rho} = 6\tfrac{2}{3}$$

$$\bar{y} = \frac{8\rho}{4\rho} = 2 = \frac{M_x}{\text{Mass}}$$

3. Find y(x), the solution to $(x^2+1)\frac{dy}{dx} = y^2$, $y(0) = 1$.

$$\int y^{-2}\,dy = \int \frac{1}{x^2+1}\,dx \quad 2$$

$$-\frac{1}{y} = \arctan x + C \quad 4$$

$$-\frac{1}{1} = \arctan 0 + C \quad 2$$

14

$$-1 = C \quad 2$$

$$-\frac{1}{y} = \arctan x - 1 \quad 2$$

$$y = \frac{-1}{\arctan x - 1} \quad 2$$

$$\boxed{y = \frac{1}{1-\arctan x}}$$

4. Let $f(x) = (x^{-8} + Ax^{-10})$ for $x \geq 1$ (for $x < 1$, $f(x) = 0$).

a. Find the value of A in order that f(x) is a probability density function.

$$1 = \int_1^{+\infty} (x^{-8} + Ax^{-10})\,dx = \lim_{b \to +\infty} \left(-\frac{1}{7}x^{-7} - \frac{1}{9}Ax^{-9}\right)\Big|_1^b \quad 2$$

14

$$= \frac{1}{7} + \frac{1}{9}A \Rightarrow \frac{6}{7} = \frac{1}{9}A \Rightarrow A = \frac{54}{7} = \boxed{7\tfrac{5}{7}}$$

(7)

b. Find the mean of the probability density function. 3

$$\mu = \int_1^{+\infty} (x^{-7} + \frac{54}{7}x^{-9})\,dx = \lim_{b \to +\infty} -\frac{1}{6}x^{-6} + \frac{54}{7(-8)}x^{-8}\Big|_1^b$$

(7)

$$= \frac{1}{6} + \frac{54}{7(8)} = \frac{31}{28}$$

$$= \frac{56 + 6(54)}{6(7)(8)} = \frac{380}{(6)(7)(8)} = \frac{95}{84} = 1 + \frac{11}{84}.$$

5. Find y(x), the solution to $\frac{dy}{dx} - \frac{\sin(x)y}{\cos(x)} = 1 \cdot y(0) = 2$.

$I(x) = e^{\int -\frac{\sin x}{\cos x}} = e^{\ln \cos x} = \cos x$ 4

$\frac{d}{dx}(y \cos x) = \cos x$

$y \cos x = \sin x + C$

$y = \frac{\sin x + C}{\cos x}$ 4

14 $2 = y(0) = \frac{0 + C}{1} \Rightarrow C = 2$ 1

$\boxed{y = \frac{\sin x + 2}{\cos x}}$ 1

6. Find the solution to $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 16y = 0$, y(0) = 1, y'(0) = 4.

$r^2 + 10r + 16 = 0$

$(r+8)(r+2)$

14 $r = -8 \quad r = -2$

$y(x) = c_1 e^{-8x} + c_2 e^{-2x}$; $y'(x) = -8c_1 e^{-8x} - 2c_2 e^{-2x}$

$1 = y(0) = c_1 + c_2$; $\quad 8 = 8c_1 + 8c_2$

$4 = y'(0) = -8c_1 - 2c_2$ $\quad 4 = -8c_1 - 2c_2$

$12 = 6c_2$

$2 = c_2$

$c_1 = 1 - c_2 = 1 - 2 = -1$

$\boxed{y(x) = -e^{-8x} + 2e^{-2x}}$

7. A biological population is growing at a rate directly proportional to the size of the population. At t = 0 hours, the population is 5 units and at t= 4 hours the population is 6 units. Find the population at t = 7 hours.

4 $P(t) = 5e^{kt}$ $\quad P(t) = 5e^{(\frac{1}{4}\ln 1.2)t}$

14 $6 = P(4) = 5e^{k \cdot 4}$ $\quad P(7) = 5 \cdot e^{\frac{7}{4}\ln 1.2}$

$\frac{6}{5} = e^{k4}$ $\quad = 5(1.2)^{7/4}$

$\ln 1.2 = k4$ $\quad = \boxed{\text{6.8839 units}}$

$k = \frac{1}{4}\ln 1.2$

6.8792 units

MAT 162 Test 3 Name Key

Fall 2018 Version 1

For full credit, show all work.

12 each
6 correct conv.
6 reasoning

1. Tell why each series is conditionally convergent, absolutely convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

$\sum_{n=1}^{\infty} \left|\frac{\sin(n)}{n^2}\right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges $p=2$

6

$\therefore$ By direct comparison $\sum_{n=1}^{\infty} \left|\frac{\sin(n)}{n^2}\right|$ converges

$\therefore \sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converges absolutely 6

more space

(b) $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

$a_n = \frac{n^2}{n^3+1}$ $b_n = \frac{1}{n}$ $\lim_{n\to+\infty} \frac{a_n}{b_n} = \lim_{n\to+\infty} \frac{n^3}{n^3+1} = 1$

6

$\therefore$ $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges ($p=1$ series)

$\therefore$ By limit comparison $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ diverges 6

(c) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\, ln(n)}$

change to 2

$a_n = \frac{1}{n \ln(n)} > 0$, $\lim_{n\to+\infty} a_n = 0$ and $a_n > a_{n+1}$

4

$\therefore$ By Alt. Series Test $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \ln(n)}$ converges.

But $\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{n \ln n}\right| = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges

4

by integral test $\int_{n+1}^{+\infty} \frac{1}{x \ln x}\,dx = \lim_{b\to+\infty} \ln(\ln x)\Big|_{n+1}^{b} = +\infty$

integral diverges forcing $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ to diverge.

4 $\therefore \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges conditionally.

2. Find the radius and interval of convergence for f(x) = $\sum_{n=0}^{\infty}(x+4)^{2n}(6+n)^{-2}$.

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty}\left|\frac{(x+4)^{2n+2}}{(6+n+1)^2}\cdot\frac{(6+n)^2}{(x+4)^{2n}}\right|$$

$$= |x+4|^2 \lim_{n\to+\infty}\left(\frac{6+n}{7+n}\right)^2 = |x+4|^2 < 1$$

5

$|x+4| < 1$

$-1 < x+4 < 1$

$-5 < x < -3$

$R = 1$

$(-5,-3)$

more space

at $x=-5$ $\sum_{n=0}^{\infty}\frac{1}{(6+n)^2}$ compare to $\sum \frac{1}{n^2}$ converges $p=2$

$\therefore \sum_{n=0}^{\infty}\frac{1}{(6+n)^2}$ converges

at $x=-3$ $\sum_{n=0}^{\infty}\frac{1}{(6+n)^2}$ converges

$\therefore$ interval of convergence is $[-5,-3]$

3. Use a power series to estimate $\int_0^{0.1} xe^{-x^3}\,dx$ with an error less than 10^{-5}.

$$e^x = \sum_{n=0}^{\infty}\frac{1}{n!}x^n$$

$$e^{-x^3} = \sum_{n=0}^{\infty}\frac{1}{n!}(-1)^n x^{3n}$$

$$x e^{-x^3} = \sum_{n=0}^{\infty}\frac{1}{n!}(-1)^n x^{3n+1}$$

$$\int_0^{0.1}\sum_{n=0}^{\infty}\frac{1}{n!}(-1)^n x^{3n+1}\,dx$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}(-1)^n \frac{x^{3n+2}}{3n+2}\Big|_0^{.1}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}(-1)^n \frac{(.1)^{3n+2}}{3n+2}$$

$$\frac{1(.1)^2}{2} - \frac{(.1)^5}{5}$$

$< 10^{-5}$

$\frac{(.1)^2}{2}$ is the answer

4. Use the fact that the following series is a telescoping series to calculate $\sum_{n=4}^{\infty}\frac{1}{n^2-5n+6}$ exactly.

12

$$\frac{1}{(n-3)(n-2)} = \frac{1}{n-3} - \frac{1}{n-2}$$

$n=4$ $\quad \frac{1}{4-3} - \cancel{\frac{1}{4-2}}$

$n=5$ $\quad \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$

$n=6$ $\quad \cancel{\frac{1}{3}} - \frac{1}{4}$

$$\lim_{n\to+\infty} S_n = 1$$

$$\boxed{\sum_{n=4}^{\infty}\frac{1}{n^2-5n+6} = 1}$$

5. Calculate $\sum_{n=3}^{\infty} e^{-4n}$ exactly.

10

$a = e^{-12}$ 3

$r = e^{-4}$

3 $\frac{a}{1-r} = \frac{e^{-12}}{1-e^{-4}}$ 1

6. Use the integral test to determine the number of terms in the partial sum for $\sum_{n=1}^{\infty} \frac{1}{n^4}$ that will estimate the infinite series with an error less than 10^{-4}

12

3 $f(x) = \frac{1}{x^4}$

3 $R_m \leq \int_m^{+\infty} \frac{1}{x^4}\,dx < 10^{-4}$

3 $\lim_{b \to +\infty} -\frac{1}{3} x^{-3} \Big|_m^b = \frac{1}{3m^3} < 10^{-4}$

$\frac{10^4}{3} < m^3$

3 $\left[\frac{10^4}{3}\right]^{\frac{1}{3}} < m$

$14.94 < m$

$\therefore \boxed{15 \leq m}$

24
19
43

MAT 162 Test 3 Name Key

Fall 2018 Version 2

For full credit, show all work.

1. Tell why each series is conditionally convergent, absolutely convergent or divergent.

12 each

(a) $\sum_{n=1}^{\infty} \frac{3^{-n}}{n^2}$

6

$\frac{3^{-n}}{n^2} \le \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p = 2 series)

$\therefore$ By Direct Comparison, $\sum_{n=1}^{\infty} \frac{3^{-n}}{n^2}$ converges

But all $a_n > 0$ $\therefore \sum_{n=1}^{\infty} \frac{3^{-n}}{n^2}$ converges absolutely 6

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$

4 $a_n = \frac{1}{n+1} > 0$, $a_n > a_{n+1}$, and $\lim_{n \to +\infty} a_n = 0$

$\therefore$ By Alt Series Test, series converges

However $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n+1}\right| = \sum_{n=2}^{\infty} \frac{1}{n}$ (p-series p = 1) diverges

4 $\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$ converges conditionally 4

(c) $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$

6

Let $f(x) = \frac{1}{x \ln x} > 0$, decreasing on $[2, +\infty]$

$\int_{n+1}^{+\infty} f(x)\,dx = \lim_{b \to +\infty} \int_{n+1}^{b} \frac{1}{x \ln x}\,dx = \lim_{b \to +\infty} \ln(\ln(b)) - \ln(\ln(n+1))$

$= +\infty$

$\therefore$ By integral test $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ diverges 6

2. Find the radius and interval of convergence for $f(x) = \sum_{n=0}^{\infty}(x+6)^{2n}(4+n)^{-2}$.

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty}\left|\frac{(x+6)^{2n+2}}{(4+n+1)^2}\cdot\frac{(4+n)^2}{(x+6)^{2n}}\right| = |x+6|^2 \lim_{n\to+\infty}\left(\frac{4+n}{5+n}\right)^2$$

(14) 5

$$= |x+6|^2 < 1$$

$\therefore$ absolute convergence for $|x+6| < 1^2$ or $-1 < x+6 < 1$ or $-7 < x < -5$ 2

$R = 1$ 2

at $x=-7$ $\sum_{n=0}^{\infty}\frac{(-1)^{2n}}{(4+n)^2} = \sum_{n=0}^{\infty}\frac{1}{n^2+8n+16} \le \sum_{n=0}^{\infty}\frac{1}{n^2}$ converges $(p=2)$

$\therefore \sum_{n=0}^{\infty}\frac{(-1)^{2n}}{(4+n)^2}$ converges $\Rightarrow$ interval of convergence

at $x=-5$, $\sum_{n=0}^{\infty}\frac{1}{(4+n)^2}$ converges is $[-7,-5]$ 2

3. Use a power series to estimate $\int_0^{0.1} xe^{-x^4}\,dx$ with an error less than 10^{-6}.

2 $e^x = \sum_{n=0}^{\infty}\frac{1}{n!}x^n$; $e^{-x^4} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{4n}$ 2 ; $xe^{-x^4} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{4n+1}$ 2

(16)

$$\int_0^{0.1} xe^{-x^4}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^{0.1} x^{4n+1}\,dx \quad 2$$

2

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{(.1)^{4n+2}}{4n+2} \quad 2 \quad = \frac{(.1)^2}{2} - \frac{(.1)^6}{6} + \cdots$$

$\frac{(.1)^6}{6} < 10^{-6}$ 2

$\boxed{\text{Answer is } \frac{(.1)^2}{2}.}$ 2

4. Use the fact that the following series is a telescoping series to calculate $\sum_{n=4}^{\infty}\frac{1}{n^2-3n+2}$ exactly.

$$\frac{1}{n^2-3n+2} = \frac{1}{(n-2)(n-1)} = \frac{1}{n-2} - \frac{1}{n-1} \quad 4$$

12

$$\sum_{n=4}^{\infty}\left(\frac{1}{n-2} - \frac{1}{n-1}\right) = \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{5}\right) + \cdots \quad 4$$

$$= \boxed{\frac{1}{2}} \quad 4$$

5. **Calculate $\sum_{n=4}^{\infty} e^{-3n}$ exactly.**

$a = e^{-3(4)}$ 3

$r = e^{-3}$ 3

10

3 $\frac{a}{1-r} = \boxed{\frac{e^{-12}}{1-e^{-3}}}$ ✓

6. **Use the integral test to determine the number of terms in the partial sum for $\sum_{n=1}^{\infty} \frac{1}{n^5}$ that will estimate the infinite series with an error less than 10^{-4}**

12

3 $f(x) = \frac{1}{x^5}$

3 $R_m \leq \int_m^{+\infty} \frac{1}{x^5}\,dx < 10^{-4}$

3 $\int_m^{+\infty} \frac{1}{x^5}\,dx = \lim_{b \to +\infty} \int_m^b \frac{1}{x^5}\,dx = \lim_{b \to +\infty} -\frac{1}{4}x^{-4}\Big|_m^b = \frac{1}{4m^4}$

3 $\frac{1}{4m^4} < 10^{-4}$

3 $\frac{10^4}{4} < m^4$

7.07 $= \frac{10}{4^{\frac{1}{4}}} < m$

$8 \leq m$