(11AM)
Fri
March 28

Ch 11 Infinite Sequences + Series

11.1 Sequences

A sequence is a list of numbers with a definite order.

$$a_1, a_2, a_3, \ldots, a_n, a_{n+1}, \ldots$$

examples:

$\{a_n\} = 5, 11, 23, 47, \ldots$

$$a_1 = 5, \quad \underbrace{a_n = 2 \cdot a_{n-1} + 1}_{\text{rule}}$$

$$\{b_n\} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots \frac{1{,}000{,}000}{1{,}000{,}001}, \ldots$$

$\uparrow$ rule

Each number in the sequence is called a <u>term</u>.

More complicated definition:

A sequence is a <u>function</u> where the domain is $\{1, 2, 3, \ldots\}$ the positive integers + the range is the terms of the sequence.
We don't use $f(x)$ or even x, we use a_n: n is the input, a_n is the output.

More notation

$$\{a_1, a_2, a_3, \ldots\} = \{a_n\} = \{a_n\}_{n=1}^{\infty}$$

↑ means the whole sequence

a_1 = 1st term
a_2 = 2nd
a_n = nth term

Limits + convergence or divergence

Defn: A sequence $\{a_n\}$ has the limit L which we write

$$\lim_{n \to \infty} a_n = L$$

if we can make the terms a_n as close to L as we want by making n sufficiently large.

$(a_n \to L \text{ as } n \to \infty)$

If the limit exists then we say the sequence converges or is convergent.
If the limit does not exist, we say the sequence diverges or is divergent.

On p. 1 of our notes:

$\{a_n\} = 5, 11, 23, 47, \ldots$ diverges

$\{b_n\} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots, \frac{1{,}000{,}000}{1{,}000{,}001}, \ldots$

converges to 1.

Further tweaking of definition of limit for sequences.

$\lim_{n\to\infty} a_n = L$

$\{a_n\}$ converges if

for every $\varepsilon > 0$ there is some integer $N > 0$ such that if $n > N$, then $|a_n - L| < \varepsilon$.

How to test for convergence ??

1st test compares the sequence $\{a_n\}$ to an ordinary function $f(x)$.

compare n & x

& compare a_n & $f(x)$.

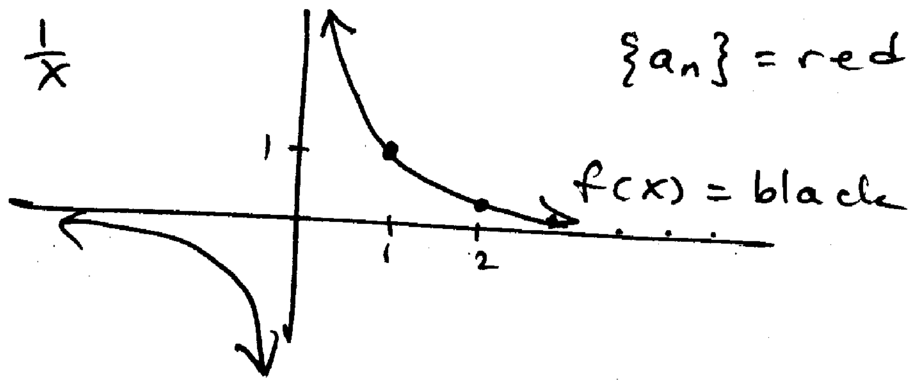
If in the function $f(x)$, $\lim_{x\to\infty} f(x) = L$

then $\lim_{n\to\infty} a_n = L$ also. example: $\rightarrow$

$\{a_n\} = \{\frac{1}{n}\} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

Let $f(x) = \frac{1}{x}$

$\{a_n\}$ = red



We know $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x} = 0.$

$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0.$

Defn: What does this mean?

$\lim_{n\to\infty} a_n = \infty$

It means $\{a_n\}$ is divergent, it diverges to ∞

So this means

for __every__ positive number M

there is some int N such that if $n > N$ then $a_n > M$.

A sequence can diverge without diverging to ∞.

$\{a_n\} = -a_n^2 = -1, -4, -9, -16, \ldots$

diverges to $-\infty$

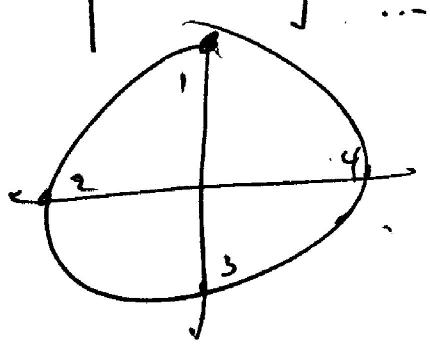
$\{b_n\} = (-n)^n = -1, +4, -27, 256, -3125, \ldots$

diverges

$\{c_n\} = \sin(n \cdot \frac{\pi}{2}) = \{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$

diverges

n	$n\frac{\pi}{2}$	$\sin(\frac{n\pi}{2})$
1	$\pi/2$	1
2	π	0
3	$3\pi/2$	-1
4	2π	0
5	$5\pi/2$	1
6	3π	0
7	$7\pi/2$	-1
8	4π	0
		$\ldots$



(5pm)

Limit Laws for Sequences

Let $\{a_n\}$ + $\{b_n\}$ be 2 convergent sequences + let c be a constant.

Then

1. $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} (a_n) + \lim\limits_{n\to\infty} (b_n)$.

2. $\lim\limits_{n\to\infty} (a_n - b_n) = \lim\limits_{n\to\infty} (a_n) - \lim\limits_{n\to\infty} (b_n)$.

3. $\lim\limits_{n\to\infty} (c \cdot a_n) = c \cdot \lim\limits_{n\to\infty} (a_n)$.

4. $\lim\limits_{n\to\infty} (c) = c$

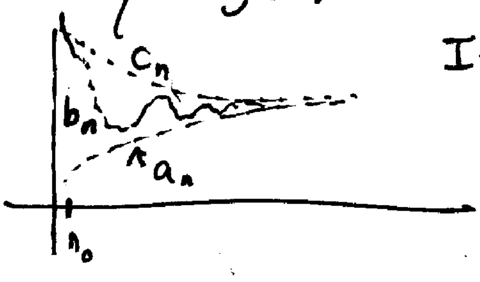
5. $\lim\limits_{n\to\infty} (a_n \cdot b_n) = \lim\limits_{n\to\infty} (a_n) \cdot \lim\limits_{n\to\infty} (b_n)$

6. $\lim\limits_{n\to\infty} \left(\dfrac{a_n}{b_n}\right) = \dfrac{\lim\limits_{n\to\infty} (a_n)}{\lim\limits_{n\to\infty} (b_n)}$ if $\lim\limits_{n\to\infty} (b_n) \neq 0$.

7. $\lim\limits_{n\to\infty} (a_n)^p = \left(\lim\limits_{n\to\infty} a_n\right)^p$ <u>if</u> $p > 0$ + $a_n > 0$.

(because $0^{\text{neg \#}}$ is undefined + $(\text{neg \#})^{\text{sq rt etc}}$ is not a real #)

Squeeze Theorem applied to sequences:



If $a_n \le b_n \le c_n$ for all $n \ge n_0$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$

then $\lim\limits_{n\to\infty} b_n = L$.

Theorem: If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.

Let $\{a_n\} = \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, -\frac{1}{25}, \frac{1}{36}, \frac{1}{49}, \frac{1}{64}, \frac{1}{81}, \frac{1}{100}, -\frac{1}{121}, \ldots$

since $\lim_{n\to\infty} |a_n| = 0$, $\lim_{n\to\infty} a_n = 0$.

Find the following limits:

① $\lim_{n\to\infty} \frac{n}{\sqrt{10+n}}$ div num + denom by $\sqrt{n}$

$$= \lim_{n\to\infty} \frac{\frac{n}{\sqrt{n}}}{\frac{\sqrt{10+n}}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{\frac{10}{n}+\frac{n}{n}}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{\frac{10}{n}+1}} = \lim_{n\to\infty} \frac{\sqrt{n}}{1}$$

$0 + 1 = 1$ $\sqrt{1} = 1$

$$= \lim_{n\to\infty} \sqrt{n} = \infty$$ (diverges)

② $\lim_{n\to\infty} \frac{\ln n}{n}$ $\frac{\infty}{\infty}$ $\therefore$ L'Hospital

$$\overset{L}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{n} = 0$$ (converges)

③ $\lim_{n\to\infty} \frac{(-1)^n}{n} = \lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0$

playing: $\{a_n\} = -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots$

(alternating sequence)

look at $|a_n| = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

$\lim_{n\to\infty} \left(\frac{1}{n}\right) = 0 \quad \therefore \lim_{n\to\infty} a_n = 0$

Theorem: If: $\lim_{n\to\infty} a_n = L$

and let f be a function is continuous at $x = L$,

then

$$\lim_{n\to\infty} f(a_n) = f(L).$$

Ex: Find $\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right)$

First look at $\frac{\pi}{n}$: $\lim_{n\to\infty} \frac{\pi}{n} = 0$

Let $f(x) = \sin(x)$. $f(x) = \sin x$ is continuous at $x = 0$.

then $\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin(0) = 0$.

Ex: Find $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right)$. $\lim_{n\to\infty} \frac{\pi}{n} = 0$, $\cos x$ is cont. @ $x = 0$.

$\therefore \lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1.$

Ex: Consider $a_n = \frac{n!}{n^n}$. Does this converge?

n	a_n
1	$\frac{1!}{1^1} = \frac{1}{1} = 1$
2	$\frac{2!}{2^2} = \frac{2}{4} = \frac{1}{2}$
3	$\frac{3!}{3^3} = \frac{\cancel{3}\cdot 2\cdot 1}{\cancel{3}\cdot 3\cdot 3} = \frac{4}{9}$
4	$\frac{4!}{4^4} = \frac{\cancel{4}\cdot 3\cdot 2\cdot 1}{\cancel{4}\cdot 4\cdot 4\cdot 4}$
n	$\frac{n!}{n^n} = \frac{n\cdot(n-1)\cdots 3\cdot 2\cdot 1}{n\cdot n\cdots n\cdot n\cdot n}$

put in opp order

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n}\underbrace{\left(\frac{2}{n}\cdot\frac{3}{n}\cdots\frac{(n-1)}{n}\cdot\frac{n}{n}\right)}_{<1 \text{ for } n<2}$$

as $n\to\infty$ this will always be <1

$$= \lim_{n\to\infty}\left(\frac{1}{n}\right)\cdot\lim_{n\to\infty}\left(\text{a number between } 0 + 1\right)$$

So a_n terms are between $\underbrace{\frac{1}{n}\cdot 0}_{=0}$ + $\underbrace{\frac{1}{n}\cdot 1}_{=\frac{1}{n}}$

We know $\lim_{n\to\infty} 0 = 0$ + $\lim_{n\to\infty}\frac{1}{n} = 0$

$\therefore$ by the squeeze theorem, $\lim_{n\to\infty} a_n = 0$.