MAT 161 Test 1 Name Key

Fall 2012

Instructions: To receive credit for all answers, show all work clearly in the space provided.

1. Find the indicated limits.

a. $\lim_{x\to -3} \dfrac{2x^2+x-15}{5x^3+17x^2+6x}$

6

$\lim_{x\to -3} \dfrac{\cancel{(x+3)}(2x-5)}{x\cancel{(x+3)}(5x+2)}$

$\dfrac{2(-3)-5}{-3(-15+2)} = \dfrac{-11}{3(13)} = \dfrac{-11}{39}$

b. $\lim_{x\to 3} \dfrac{2x^2+x-15}{5x^3+17x^2+6x}$

6

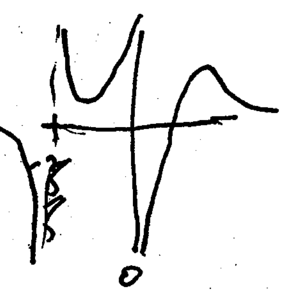
$\lim_{x\to 3} \dfrac{2x-5}{x(5x+2)} = \dfrac{2(3)-5}{3(17)} = \dfrac{1}{51}$

c. $\lim_{x\to 0^+} \dfrac{2x^2+x-15}{5x^3+17x^2+6x} = -\infty$

6

$5x+2$	$-$	$+$	$+$	$+$
x	$-$	$-$	$+$	$+$
$2x-5$	$-$	$-$	$-$	$+$
	$(-)$	$(+)$	$(-)$	$(+)$

$-\frac{2}{5}$, 0, $\frac{5}{2}$



d. $\lim_{x\to +\infty} \dfrac{2x^2+x-15}{5x^3+17x^2+6x}$

6

$= \lim_{x\to +\infty} \dfrac{\frac{2}{x} + \frac{1}{x^2} - \frac{15}{x^3}}{5 + \frac{17}{x} - \frac{6}{x^2}}$

$= 0$

2. Suppose that for all real numbers x between 0 and π, $2\sin(x) \le f(x) \le 1+\sin^2(x)$.

a. Show that f(x) is continuous at x= $\pi/2$.

10

$\lim_{x\to \frac{\pi}{2}} 2\sin x = 2(1) = 2$

$\lim_{x\to \frac{\pi}{2}} 1+\sin^2(x) = 1+1 = 2$

$\Rightarrow$ $f(\pi/2) = 2$ and $\lim_{x\to\frac{\pi}{2}} f(x) = 2$ by Squeeze Law

$\therefore$ f is continuous at $\frac{\pi}{2}$. 2

b. Is f(x) continous at x = $\pi/6$? Why or why not?

5

at $\frac{\pi}{6}$

$2\sin(\frac{\pi}{6}) \le f(\frac{\pi}{6}) \le 1+\sin^2(\frac{\pi}{6})$

$2(\frac{1}{2}) \le f(\frac{\pi}{6}) \le 1+(\frac{1}{2})^2$

$1 \le f(\frac{\pi}{6}) \le 1.25$

We dont have enough information to conclude f is continuous at $\frac{\pi}{6}$.

2

/33

3. Using the precise (δ,ε) definition of limits, prove that $\lim_{x\to 3}(2-7x)=-19$

12

Given $\epsilon>0$ choose $\delta = \epsilon/7$.

if $0<|x-3|<\delta$, then $|x-3|<\epsilon/7$

$|7x-21|<\epsilon$

$|21-7x|<\epsilon$

$|2-7x-(-19)|<\epsilon$.

4. Use the Intermediate Value Theorem to show that there is a solution to the equation $10=(x-3)^2+7x$ in the interval $(-1,3)$.

12

Let $p(x)=(x-3)^2+7x-10$ is continuous since $p(x)$ is a polynomial.

$p(-1)=(-1-3)^2+7(-1)-9=(-4)^2-7-10=-1$

$p(3)=0^2+7(3)-10=11$

0 is a number between -1 & 11

$\therefore$ there is a z between -1 & 3 such that $p(z)=0$

z is a solution to $(z-3)^2+7x-10=0$

$\therefore\ 10=(z-3)^2+7z$.

5. Let $f(x)=(3x-5)^2$. Find the equation of the tangent line to the curve y = f(x) at (2, 1).

15

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(3x+3h-5)^2-(3x-5)^2}{h}$$

$$=\lim_{h\to 0}\frac{(3x-5+3h)^2-(3x-5)^2}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{(3x-5)^2}+2(3x-5)3h+9h^2-\cancel{(3x-5)^2}}{h}$$

$$=\lim_{h\to 0}2(3x-5)3+9h=6(3x-5)$$

$f'(2)=6(3\cdot 2-5)=6$

$y-1=6(x-2)$

$y-1=6x-12$

$y=6x-11$.

39

6. Suppose a stone is thrown vertically upward with an initial velocity of 32 ft/s from a bridge 16 feet above a river. By Newton's Laws of Motion, the position of the stone measured as the height above the river after t seconds is s(t) = $-16t^2 + 32t + 16$ where s = 0 is the level of the river.

   a. Find s'(t).

8

$$s'(t) = \lim_{h\to 0} \frac{s(t+h) - s(t)}{h} = \lim_{h\to 0} \frac{-16(t+h)^2 + 32(t+h) + 16 - (-16t^2 + 32t + 16)}{h}$$ 2 2

$$= \lim_{h\to 0} \frac{-16(\cancel{t^2} + 2th + h^2) + \cancel{32t} + 32h + \cancel{16} - (\cancel{-16t^2} + \cancel{32t} + \cancel{16})}{h}$$ 1

$$= \lim_{h\to 0} -16(2t + h) + 32$$ 1

$$= -32t + 32$$ 2

   b. Find the average velocity of the stone from t = 1 to t = 1.2 seconds.

6

$$\frac{s(1.2) - s(1)}{.2} = -16(2(1) + .2) + 32$$ 2 2

$$= -16(.2) = -3.2 \text{ ft/second}$$ 2

   c. Find the instantaneous velocity when t = 1 second.

6

$$s'(1) = -32(1) + 32 = 0$$ 2 2 2

   d. Find the instantaneous velocity when the stone hits the river.

8

$$-16(t^2 - 2t - 1) = 0$$ 1

$$t^2 - 2t + 1 = 2$$

$$(t-1)^2 = 2$$

$$t = 1 + \sqrt{2}$$ 2

$$s'(1+\sqrt{2}) = -32(1+\sqrt{2}) + 32 = -32(\sqrt{2}) \text{ ft/sec.}$$ 2 1 2

$$= -45.25 \text{ ft/sec}$$

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MAT 161
Fall 2012

Test 2

Name Key

Instructions: To receive credit for all answers, show all work clearly in the space provided.

1. Let $f(x) = 5x^4 - 4\sqrt{5x+20} + 21x$

   a. Find $f'(x)$.

$$f'(x) = 20x^3 - 4\left(\tfrac{1}{2}\right)(5x+20)^{-\frac{1}{2}}(5) + 21$$
$$= 20x^3 - 10(5x+20)^{-\frac{1}{2}} + 21$$

5 $\frac{1}{2}$ point per mark

   b. Find the equation of the tangent line to the curve $y = f(x)$ at (1, 6).

5

$$f'(1) = 20 - 10(25)^{-\frac{1}{2}} + 21 = 41 - \frac{10}{5} = 39 \quad 2$$
$$y - 6 = 39(x-1)$$
or $y = 39x - 33$

2. Find the second derivative of $f(x) = \ln(5 + \cos(3x))$.

(10)

$$f'(x) = \frac{1 \cdot 1}{(5+\cos(3x))} \cdot (-\sin(3x)) \cdot 3 = \frac{-3\sin(3x)}{5+\cos(3x)} = -3\left[\frac{\sin(3x)}{5+\cos 3x}\right]$$

$$f''(x) = \frac{(5+\cos(3x))(-3\cos(3x)(3)) - (-3\sin(3x))(0-\sin(3x)(3))}{(5+\cos(3x))^2}$$

$$= \frac{-9\left[(5+\cos(3x))\cos(3x) + \sin^2(3x)\right]}{(5+\cos(3x))^2} = \frac{-9\left[5\cos(3x)+1\right]}{\left[5+\cos(3x)\right]^2}$$

3. Calculate the linearization of $f(x) = x^{\frac{2}{3}}$ at a = 1000 Then estimate $(1000.15)^{\frac{2}{3}}$ using the linearization.

(10)

$f(x) = x^{2/3}$ $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$

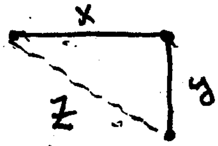
$f(1000) = 1000^{2/3} = 100$ $f'(1000) = \frac{2}{3}\frac{1}{10} = \frac{1}{15}$

$$L(x) = f(1000) + f'(1000)(x-1000)$$
$$= 100 + \frac{1}{15}(x-1000)$$

$$L(1000.15) = 100 + \frac{1}{15}(.15) = 100 + \frac{1}{100} = 100.01$$

4. Before takeoff two jets on perpendicular runways are converging toward the intersections of the runways. Jet A is moving at a rate of 10 miles per hour and jet B is moving at a rate of 15 miles per hour when each is ¼ mile from the intersection. At what rate is the distance between the jets changing?

10



$\frac{dz}{dt} = ?$ when $\frac{dx}{dt} = -10$ $\quad \frac{dy}{dt} = -15$

$x = \frac{1}{4} \quad y = \frac{1}{4}$

$z^2 = x^2 + y^2$

$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

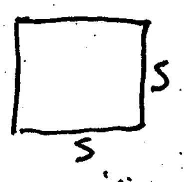
$\frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z}$

½ point per vertical bar

$= \frac{\frac{1}{4}(-10) + \frac{1}{4}(-15)}{\sqrt{(\frac{1}{4})^2 + (\frac{1}{4})^2}} = \frac{-10 - 15}{\sqrt{2}} = \frac{-25}{\sqrt{2}} \approx -17.68$ miles/hr.

5. The perimeter of a square is measured at 1000 ± 3 cm. Calculate the area with an estimate for the error using differentials..

10



$P = 4s$

$A = s^2 = \left(\frac{P}{4}\right)^2 = \frac{1}{16}P^2$

$dA = \frac{1}{16}2P\,dP = \frac{1}{8}P\,dP$

1 point per bar

area $A = \frac{1}{16}(1000)^2 = 62500 \text{ cm}^2$

estimate $dA \approx \frac{1}{8}(1000)(3) = 375 \text{ cm}^2$

In problems 6 – 10, calculate the derivative of y with respect to x.

6. $y = \dfrac{5x+3^{2x}}{\cos x}$

(10) $\dfrac{dy}{dx} = \dfrac{(\cos x)(5 + 3^{2x}(\ln 3)2) - (5x+3^{2x})(-\sin x)}{(\cos x)^2}$

15 marks
$\frac{1}{2}$ point per mistake

7. $y = \sin(\ln x) + (\cos 8x)^2$

(10) $\dfrac{dy}{dx} = \cos(\ln x)\left(\frac{1}{x}\right) + 2\cos(8x)(-\sin(8x))(8)$

1 point per mistake

8. $y = \sin(\arctan x)$

(10) $\dfrac{dy}{dx} = \cos(\arctan x)\left(\dfrac{1}{1+x^2}\right)$

2 points per mark

9. $6x^2 - xy^2 + 5y^3 = 3x - 4y$

(10) $12x - 1y^2 - x2y\dfrac{dy}{dx} + 15y^2\dfrac{dy}{dx} = 3 - 4\dfrac{dy}{dx}$

$(-2xy + 15y^2 + 4)\dfrac{dy}{dx} = 3 - 12x + y^2$

$\dfrac{dy}{dx} = \dfrac{3-12x+y^2}{-2xy+15y^2+4}$

1 point per mark

10. $y = (1+x^5)^{(2x+4)}$

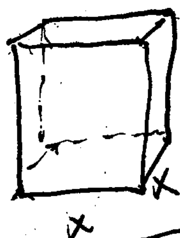
(10) $\ln y = (2x+4)\ln(1+x^5)$

$\dfrac{dy}{dx} = y\dfrac{d\ln y}{dx} = (1+x^5)^{2x+4}\left\{2\ln(1+x^5) + (2x+4)\dfrac{1}{1+x^5}5x^4\right\}$

1 point per mark

MAT 161 **Test 3** **Name** Key
Fall 2012
Instructions: To receive credit for all answers, show all work clearly in the space provided.

1. Find the dimensions of a 500 cubic foot rectangular steel holding tank, square-based, open-top, that is of minimal surface area.



$V = 500 = x^2 h \Rightarrow h = 500x^{-2}$

min $S = x^2 + 4xh = x^2 + 2000x^{-1}$ 6

min $f(x) = x^2 + 2000x^{-1}$ on $(0, +\infty)$

15

$f'(x) = 2x - 2000x^{-2} = 0$

$2x = \frac{2000}{x^2}$

$x^3 = 1000$

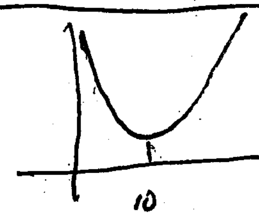
$x = 10$

$f'(10) = 0$

$f''(x) = 2 + 4000x^{-3}$

$f''(10) > 0$

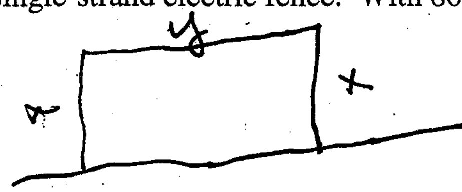
f min at $x = 10$



6

f min when $\boxed{\begin{array}{l} x = 10 \\ h = 5 \end{array}}$ 3

2. A rectangular plot of farm land will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 meters of wire, what is the largest area you can enclose.



$2x + y = 800$ 6

$y = 800 - 2x$

Max $xy = 800x - 2x^2$ on $[0, 400]$

15

$f(x) = 800x - 2x^2$ on

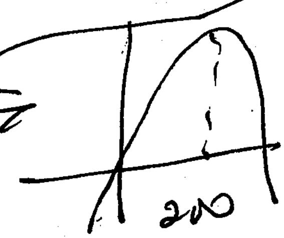
$f'(x) = 800 - 4x = 0$

$x = 200$

$f'(200) = 0$

$f''(x) = -4$

f has a rel max at $x = 200$



6

max Area is 3

$200(800 - 2(200))$

$(200)(400) = \boxed{80{,}000}$

/30

3. Calculate $\lim_{x\to 0} \frac{x-\sin(x)}{4x^3}$.

(10)

$$\lim_{x\to 0} \frac{1-\cos x}{12x^2} \quad 3$$

$$\lim_{x\to 0} \frac{\sin x}{24x} \quad 3$$

$$\lim_{x\to 0} \frac{\cos x}{24} = \boxed{\frac{1}{24}} \quad 3 \quad 1$$

4. Find the rel. max of $y = f(x) = (x-30)^2e^{-4x}$.

(10)

$$f'(x) = 2(x-30)e^{-4x} + (x-30)^2(-4)e^{-4x} \quad 2$$

$$= 2(x-30)e^{-4x}[1-2(x-30)]$$

$$= 2(x-30)e^{-4x}[-2x+61]$$

$$= 2(x-30)(-2)e^{-4x}[x-\tfrac{61}{2}]$$

$$= -4(x-30)e^{-4x}[x-\tfrac{61}{2}] \quad 3$$

$-4e^{4x}$: $-$ $-$ $-$

$x-30$: $-$ $+$ $+$

$x-\frac{61}{2}$: $-$ $-$ $+$

$(-)$ 30 $(+)$ $\frac{61}{2}$ $(-)$ 3

f has a rel. max at $x = \frac{61}{2}$ 2

$f(61/2) = \frac{1}{4}e^{-122}$

$= 2.59 \times 10^{-54}$

5. Use an initial guess of 10 and Newton's Method once to estimate the solution to $x^3 - 999 = 0$.

(10)

$$x - \frac{x^3-999}{3x^2} = 10 - \frac{1000-999}{3(10^2)} = 10 - \frac{1}{300}$$

4 4 2

6. Calculate the derivative of $\sinh(x) + \cosh(2x)$.

(10)

$$\boxed{\cosh x + \sinh(2x)(2)}$$

4 1 4 1

7. For $f(x)=x^3(x-5)^2$ $= x^3(x^2-10x+25) = x^5-10x^4+25x^3$

$f'(x) = 5x^4 - 40x^3 + 75x^2$ $\quad f''(x) = 20x^3 - 120x^2 + 150x = 10x(2x^2-12x+15)$

$= 20x(x^2-6x+\frac{15}{2}) = 20x((x-3)^2-\frac{3}{2})$

a. Calculate the first and second derivative of $f(x)$.

$f'(x) = 3x^2(x-5)^2 + x^3\,2(x-5) = x^2(x-5)[3(x-5)+2x] = x^2(x-5)(5x-15)$

$= 5x^2(x-5)(x-3)$

5

$f''(x) = 10x(x-5)(x-3) + 5x^2(x-3) + 5x^2(x-5)$

$= 5x(2(x-5)(x-3) + x^2-3x + x^2-5x)$

$= 5x(2x^2-16x+30+x^2-3x+x^2-5x) = 5x(4x^2-24x+30)$

$= \cancel{10x(3x^2-8x+5)} \cancel{-10x(x-5)(x-1)}$

b. Find the intervals where $f(x)$ is increasing and decreasing. $= 5x(2x^2-12x+15)$

$= \cancel{30x} - 10x(2x\ \ )(x$

$5x^2$: + + + +

$x-1$: − − + +

$x-5$: − − − +

(+) 0 (+) 3 (−) 5 (+)

5

f is dec on $(3,5)$

f is inc on $(-\infty,0)\cup(0,3)\cup(5,+\infty)$

c. Find the intervals where $f(x)$ is concave up and concave down.

− + − +

$0 \qquad 3-\sqrt{\frac{3}{2}} \qquad 3+\sqrt{\frac{3}{2}}$

5

d. Identify the local maximum, local minimum, and inflection points.

Those — $(0,0)$, $(3-\sqrt{\frac{3}{2}}, 58.2)$ $(3+\sqrt{\frac{3}{2}}, 45.3)$ inflection points

5

$(3,108)$ is a local max.

$(5,0)$ is a local min.

e. Find the x- and y-intercepts of $y=f(x)$.

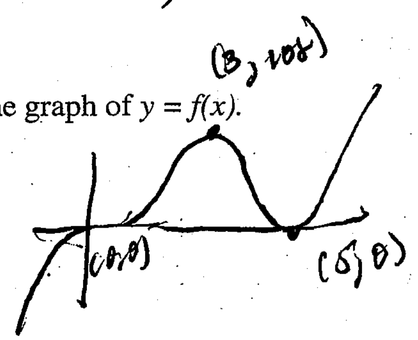
$f(0)=0$ $\quad (0,0)$ is the y intercept

5

$(0,0)$, $(5,0)$ are the x-intercepts.

f. Sketch the graph of $y=f(x)$.

5



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MAT 161 Test 4 Name Key

Fall, 2012

Show all work for credit purposes.

1. Evaluate the Riemann sum for $f(x)=3x^2-x$ on $1\le x\le 3$, with four subintervals, taking the sample points to be the left endpoints.

10

1 1.5 2 2.5 3

$\Delta x = \frac{3-1}{4} = \frac{1}{2}$

$f(1)\frac{1}{2} + f(1.5)\frac{1}{2} + f(2)\frac{1}{2} + f(2.5)\frac{1}{2} = (2+5.25+10+16.25)\frac{1}{2}$

$= (33.5)(\frac{1}{2})$

$= 16.75$

2. Suppose that $f'(x)=8x^3-\frac{1}{3}x^{-\frac{2}{3}}+2$, $f(1)=5$. Find $f(x)$.

10

$f(x) = 8\frac{x^4}{4} - \frac{1}{3}\frac{x^{\frac{1}{3}}}{\frac{1}{3}} + 2x + C$

$= 2x^4 - x^{\frac{1}{3}} + 2x + C$

$5 = f(1) = 2-1+2+C = 3+C$

$2 = C$

$\boxed{f(x) = 2x^4 - x^{\frac{1}{3}} + 2x + 2}$

3. Calculate $\lim_{n\to\infty}\sum_{k=1}^{n}(2+k\frac{4}{n})^2(\frac{4}{n})$ by evaluating the equivalent integral.

10

$b-a=4$

$a=2$

$b=6$

$\int_2^6 x^2\,dx = \frac{1}{3}x^3\Big|_2^6$

$= \frac{1}{3}6^3 - \frac{1}{3}2^3 = \frac{208}{3}$

4. Find the area from $x=1$ to $x=3$, between the x-axis and the curve $y=4x-\frac{1}{x^2}$.

10

$\int_1^3 (4x - x^{-2})\,dx = (2x^2 + x^{-1})\Big|_1^3$

$= (2(9)+\frac{1}{3}) - (2+1) = 15+\frac{1}{3}$

/40

5. Find the average value of $f(x) = 2x + \sin(x)$ on the interval $[0, \pi]$.

10

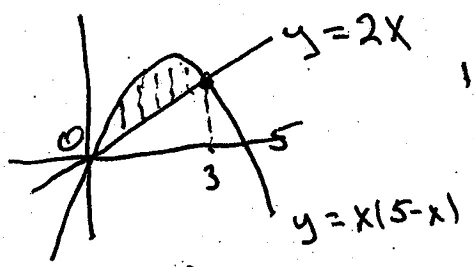
$$\frac{1}{\pi - 0}\int_0^{\pi}(2x + \sin x)\,dx = \frac{x^2 - \cos x}{\pi}\Big|_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi^2 - \cos\pi - (0^2 - \cos 0)\right]$$

$$= \frac{1}{\pi}\left(\pi^2 - (-1) + 1\right] = \frac{1}{\pi}\left[\pi^2 + 2\right] \approx 3.78$$

6. Calculate the area bounded by the curves $y = x(5 - x)$ and $y = 2x$.

10



$$\int_0^3 (x(5-x) - 2x)\,dx$$

$$\int_0^3 (3x - x^2)\,dx$$

$$\left(\frac{3}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^3$$

$$\frac{27}{2} - \frac{27}{3} = \frac{1}{6}(27)$$

$$= \boxed{\frac{9}{2}}$$

$x(5-x) = 2x$

$x(5-x) - 2x = 0$

$x[5 - x - 2] = 0$

$x[3 - x] = 0$ $\quad x = 0$, $x = 3$

7. A. Write the integral that is the volume of the solid formed by rotating about the X axis the region in question 6. You do not have to evaluate the integral.

8

$$V = \int_0^3 \pi\left([x(5-x)]^2 - (2x)^2\right]dx$$

B. Write the integral that is the volume of the solid formed by rotating about the Y axis the region in question 6. You do not have to evaluate the integral.

8

$$V = \int_0^3 2\pi x\left(x(5-x) - 2x\right)dx$$

36

8. Calculate the following.

8 a. $\int\left(\frac{4}{x^2+1}+e^{3x}\right)dx$ $= \int \frac{4}{x^2+1}\,dx + \int e^{3x}\,dx = \boxed{4\arctan x + \frac{1}{3}e^{3x} + C}$

2 2 1 2 1

$\int \frac{4}{x^2+1}\,dx = 4\arctan x + C$

$\int e^{3x}\,dx = \int e^{u}\,\frac{1}{3}\,du = \frac{1}{3}e^{u} + k$

$u = 3x$ $\quad = \frac{1}{3}e^{3x} + k$

$\frac{du}{dx} = 3 \Rightarrow \frac{1}{3}du = dx$

8 b. $\int_1^2 (2x-2)^4\,dx$ $= \int_0^2 u^4\,\frac{1}{2}\,du$

$u = 2x-2$

$\frac{du}{dx} = 2$

$\frac{1}{2}du = dx$

$= \frac{1}{2}\frac{u^5}{5}\Big|_0^2$

$= \boxed{\frac{1}{10}(2^5)}$

9. Calculate the derivative of $\int_1^{5x}(2t+352)^{10}\,dt$

8 $y = \int_1^{u}(2t+352)^{10}\,dt$ where $u = 5x$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (2u+352)^{10}(5)$

$= \boxed{(10x+352)^{10}(5)}$

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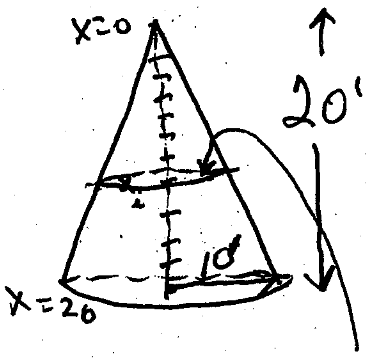
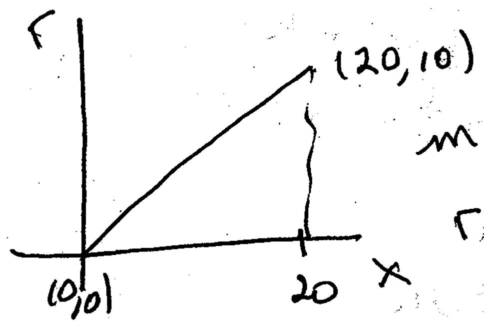
Extra Work Problem

"An underground storage tank is in the form of a cone with the tip at ground level and the base of radius 10 feet at a depth of 20 feet. Suppose the tank is filled with water. How much work does it take to pump the water to the top.





$m = \frac{10}{20} = \frac{1}{2}$

$r = \frac{1}{2}x$

$r = \frac{1}{2}x_i$

$A(x_i) = \pi\left(\frac{1}{2}x_i\right)^2 = \frac{\pi}{4}x_i^2$ area

$\Delta x\, A(x_i) = \frac{\pi}{4}x_i^2\,\Delta x$ volume

$62.5\,\Delta x\, A(x_i) = 62.5\,\frac{\pi}{4}x_i^2\,\Delta x$ weight

$x_i\, 62.5\,\frac{\pi}{4}x_i^2\,\Delta x$ work

$\lim_{n\to+\infty}\sum_{i=1}^{n} x_i^3\, 62.5\,\frac{\pi}{4}\,\Delta x$ total work approx

$$\text{work} = \int_0^{20} x^3\, 62.5\left(\frac{\pi}{4}\right)dx$$

$$= \frac{1}{4}x^4(62.5)\left(\frac{\pi}{4}\right)\Big|_0^{20}$$

$$= \frac{1}{4}(20^4)\,62.5\left(\frac{\pi}{4}\right) \text{ ft-lbs work}$$

$$= 1{,}963{,}495.4 \text{ ft-lbs.}$$