Test 2
MAT 162

Spring, 2013 Name: Key
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Directions: Show all work for partial credit purposes. You may use a graphing calculator. The test is closed book.

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1. Use Euler's Method to approximate y(2.2) given $\frac{dy}{dx} = 4x - 5y - 1$, and $y(2) = 3$. Use a stepsize of 0.1. $f(x,y)h$

x	y
2	3
2.1	2.2
2.2	1.84

$(8-15-1).1 = -.8$

$(8.4-11-1)(.1) = -.36$

$\boxed{1.84}$

2. A. Find the equilibrium solutions to $\frac{dy}{dx} = (y+1)(y-5)^2$.

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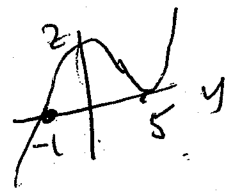
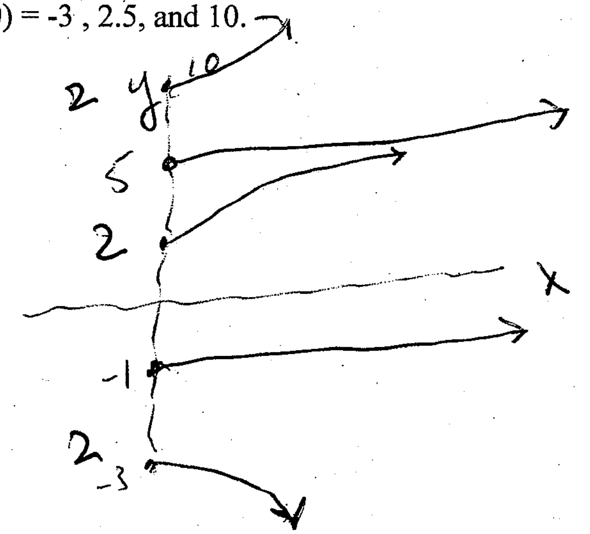
3 $y=-1$

3 $y=5$

B. On a single graph, sketch three solutions to $\frac{dy}{dx} = (y+1)(y-5)^2$ for the three differential initial conditions: $y(0) = -3$, 2.5, and 10.

$z=(y+1)(y-5)^2$





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3. Find y(x), the solution to $\frac{dy}{dx} = -3x^{-1}y+x-2$, $y(1) = 0$.

(11)

$$\frac{dy}{dx} + \frac{3}{x}y = x\cancel{+}2$$

$$I(x) = e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$$

$$\frac{d}{dx}(yx^3) = x^4 \cancel{+} 2x^3$$

$$yx^3 = \frac{1}{5}x^5 \cancel{+} \frac{2}{4}x^4 + C$$

$$y = \frac{1}{5}x^2 \cancel{+} \frac{1}{2}x + Cx^{-3}$$

$$0 = \frac{1}{5} \cancel{+} \frac{1}{2} + C = \cancel{\tfrac{7}{10}} \ -3/10 + C$$

$$C = \frac{3}{10}$$

$$\boxed{y = \frac{1}{5}x^2 - \frac{1}{2}x + \frac{3}{10}x^{-3}}$$

4. Find y(x), the solution to $\frac{dy}{dx} = x^2(1+y)^3$, $y(0) = 2$.

(11)

$$\int \frac{1}{(1+y)^3}dy = \int x^2 dx$$

$$\frac{1}{-2}(1+y)^{-2} = \frac{1}{3}x^3 + C$$

$$-\frac{1}{2}(3)^{-2} = C$$

$$-\frac{1}{2}(1+y)^{-2} = \frac{1}{3}x^3 - \frac{1}{2}3^{-2}$$

$$(1+y)^{-2} = -\frac{2}{3}x^3 + \frac{1}{9}$$

$$(1+y)^2 = \frac{1}{\frac{1}{9} - \frac{2}{3}x^3}$$

$$1+y = \sqrt{\frac{1}{\frac{1}{9} - \frac{2}{3}x^3}}$$

$$\boxed{y = -1 + \frac{1}{\sqrt{\frac{1}{9} - \frac{2}{3}x^3}}}$$

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$\frac{5}{6}(3)$

11 5. A tank is filled with 600 liters of contaminated water containing 3 kg of toxins. Pure water containing a concentration of .001 kg of toxins per liter is pumped in at a rate of 50 l/min., mixes instantaneously, and then is pumped out at the rate of 60 l/min. Find y(t) the number of grams of the toxin in the tank t minutes after the rinse begins. Then find the time at which there is .1 kg of toxin present.

$y(0) = 3$

$\frac{dy}{dt} = .001(50) - \frac{y}{600-10t}$ 60

$\frac{dy}{dt} = .05 - \frac{6y}{60-t}$

$\frac{dy}{dt} + \frac{6}{60-t} y = .05$

$I(t) = e^{\int \frac{6}{60-t}dt} = e^{-6\ln(60-t)} = (60-t)^{-6}$

$\frac{d}{dt}\left(y(60-t)^{-6}\right) = .05(60-t)^{-6}$

$y(60-t)^{-6} = \frac{.05}{5}(60-t)^{-5} + c = .01(60-t)^{-5} + c$

$y(t) = .01(60-t) + c(60-t)^{6}$

$3 = y(0) = .01(60) + c(60^6)$

$2.4 = c\,60^6 \Rightarrow c = \frac{2.4}{60^6}$

$\boxed{y(t) = .6\left(1-\frac{t}{60}\right) + 2.4\left(1-\frac{t}{60}\right)^{6}}$

Solve $y(t) = .1$

$t = 50.005128$ minutes

by calculator.

11 6. First find the solution to $\frac{d^2y}{dx^2} - 36y = 0$, $y(0) = 1$, $y'(0) = 2$.

$r^2 - 36 = 0$

$r = \pm 6$

$y(x) = c_1 e^{6x} + c_2 e^{-6x}$; $1 = y(0) = c_1 + c_2$

$y'(x) = 6c_1 e^{6x} - 6c_2 e^{-6x}$ $2 = y'(0) = 6c_1 - 6c_2$

$6 = 6c_1 + 6c_2$

$2 = 6c_1 - 6c_2$

$8 = 12c_1$

$c_1 = \frac{8}{12} = 2/3$

$c_2 = 1 - c_1 = \frac{1}{3}$

$\boxed{y(x) = \frac{2}{3}e^{6x} + \frac{1}{3}e^{-6x}}$

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7. Find the value of k so that $f(x) = x^{-10} + kx^{-4}$ is a probability density function on $[1,+\infty)$ and then find the value of the mean for the probability density function.

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$$1 = \int_1^{+\infty} (x^{-10} + kx^{-4})\,dx = \lim_{b\to+\infty} \frac{x^{-9}}{-9} + \frac{k}{-3}x^{-3}\Big|_1^b$$

$$= \frac{1}{9} + \frac{k}{3} \Rightarrow \frac{8}{9} = \frac{k}{3} \Rightarrow k = \boxed{\frac{8}{3}}$$

$$\int_1^{+\infty} x\left(x^{-10} + \frac{8}{3}x^{-4}\right)dx = \int_1^{+\infty}\left(x^{-9} + \frac{8}{3}x^{-3}\right)dx$$

$$= \lim_{b\to+\infty} \frac{x^{-8}}{-8} + \frac{8}{3}\left(\frac{1}{-2}x^{-2}\right)\Big|_1^b$$

$$= \frac{1}{8} + \frac{4}{3} = \boxed{\frac{35}{24}} = \text{mean}$$

8. For $f(x) = 17 + 6x^{1.5}$, find the length of the curve $y = f(x)$ from $x = 1$ to 5.

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$$f'(x) = 6(1.5)x^{\frac{1}{2}} = 9x^{\frac{1}{2}}$$

$$L = \int_1^5 \sqrt{1+(f'(x))^2}\,dx = \int_1^5 \sqrt{1+81x}\,dx$$

$$= (1+81x)^{3/2}\,\frac{2}{3}\,\frac{1}{81}\Big|_1^5$$

$$= \frac{2}{243}\left[406^{3/2} - 82^{3/2}\right]$$

9. A cylindrical barrel of radius 3 feet is lying on its side submerged in water 10 feet deep. (Recall that water weighs weigh2 62.5 lbs. per cubic foot.) Find the hydrostatic force against one end of the barrel.

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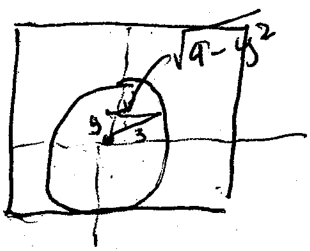

$$\int_{-3}^{3} 62.5(7-y)\,2\sqrt{9-y^2}\,dy$$

$$= \int_{-3}^{3} 62.5(7)(2)\sqrt{9-y^2}\,dy$$

$$= 62.5(7)(\cancel{2})\,\frac{\pi}{\cancel{2}}\,(3^2) = 3937.5\pi \text{ lbs.}$$

$$= 12370.02 \text{ lbs}$$

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