MAT 161 Final Exam
Thursday, June 20, 2013

Name Key

1. Calculate the following limits:

a. $\lim_{x\to 4} \frac{3x^2-8x-16}{x^2+x-20}$

10

3 $\lim_{x\to 4} \frac{\cancel{(x-4)}(3x+4)}{\cancel{(x-4)}(x+5)}$ 3

2 $\frac{3\cdot 4+4}{4+5} = \boxed{\frac{16}{9}}$ 2

b. $\lim_{x\to +\infty} \frac{20x^2}{5x+7} - 4x$

10

$\lim_{x\to +\infty} \frac{-28x}{5x+7}$ 5

$-\frac{28}{5}$ 5

2. Use the definition of the derivative to find $f'(x)$ when $f(x) = 3x^2 - 5x + 2$.

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$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ 2

$= \lim_{h\to 0} \frac{3(x+h)^2 - 5(x+h) + 2 - (3x^2-5x+2)}{h}$ 2

$= \lim_{h\to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{5x} - 5h + \cancel{2} - \cancel{3x^2} + \cancel{5x} - \cancel{2}}{h}$ 2

$= \lim_{h\to 0} (6x + 3h - 5) = 6x - 5$
2 2

3. Give a delta-epsilon argument that $\lim_{x\to 2} 5x-7 = 3$.

10

Given $\epsilon > 0$, choose $\delta = \epsilon/5$. 1

Then if $0 < |x-2| < \delta$, $|x-2| < \epsilon/5$ 2
2

$|5x-10| < \epsilon$ 2

$|5x-7-3| < \epsilon$. 2

4. Tell why $f(x) = |x^2-25|$ is not differentiable at $x = 5$ but is continuous there.

10

$\lim_{h\to 0} \frac{|(5+h)^2-25| - |5^2-25|}{h} = \lim_{h\to 0} \frac{|10h+h^2|}{h} = \lim_{h\to 0} \frac{|h|}{h}|10+h|$ 5

2 $\boxed{f \text{ not diff}}$ since the limit above does not exist!

but $\lim_{x\to 5} f(x) = 0 = f(5)$ $\therefore$ $\boxed{f\text{'s cont.}}$ there. 5
1 1 1 2

/5

5. Calculate $\frac{dy}{dx}$ for the following:

a. $y = \left[(x^2+3x)e^{4x}+x\right]^{10}$

10 $\frac{dy}{dx} = 10\left[(x^2+3x)e^{4x}+x\right]^9\left[(2x+3)e^{4x}+(x^2+3x)e^{4x}4+1\right]$

2 2 1 2 1 1 1 1 1

b. $y = \dfrac{\cos(7x)-5x}{x^{\frac{2}{3}}+10}$

10 $\frac{dy}{dx} = \dfrac{(x^{2/3}+10)(-\sin(7x)(7)-5)-(\cos(7x)-5x)(\frac{2}{3}x^{-\frac{1}{3}})}{(x^{2/3}+10)^2}$ 1

2 1 1 2 1 1 1

c. $y = 2^{7x}+\tan^{-1}(3x)$

10 $\frac{dy}{dx} = 2^{7x}(\ln 2)7 + \dfrac{1}{1+(3x)^2}$ 3 2

2 2 1 1 2

d. $y=(3x+4)^{\tan(9x)}$; $\ln y = \tan(9x)\ln(3x+4)$

10 $\frac{dy}{dx} = (3x+4)^{\tan(9x)}\left[\sec^2(9x)9\ln(3x+4)+\tan(9x)\frac{3}{3x+4}\right]$

1 1 1 2 1 1 1 2

e. $x^3+y\sin x=(y+3x)^2$

$3x^2+\frac{dy}{dx}\sin x+y\cos x = 2(y+3x)\left(\frac{dy}{dx}+3\right)$

10 $\left[\sin x - 2(y+3x)\right]\frac{dy}{dx} = 2(y+3x)(3)-3x^2-y\cos x$ 1

$\frac{dy}{dx} = \dfrac{2(y+3x)(3)-3x^2-y\cos x}{\sin x - 2(y+3x)}$ 1

f. $y = \displaystyle\int_0^{\sin(x)+cox(x)}(t^2+2)dt$

10 $y=\int_0^u (t^2+2)dt$ 1 $u=\sin x+\cos x$ 1

$\frac{dy}{du}\frac{du}{dx} = (u^2+2)(\cos x-\sin x)$

$= \left[(\sin x+\cos x)^2+2\right](\cos x-\sin x)$

1 1 1 1

Actual $\to 5.002666 5246$

6. Use a linear approximation to estimate $125.2^{\frac{1}{3}}$

10

$f(x) = x^{\frac{1}{3}}$ $\qquad f'(x) = \frac{1}{3}x^{-2/3}$

$f(125) = 5$ $\qquad f'(125) = \frac{1}{3\,5^2} = \frac{1}{75}$ 1

$L(x) = 5 + \frac{1}{75}(x - 125)$

$L(125.2) = 5 + \frac{1}{75}(.2) = 5 + \frac{1}{75(5)} = 5 + \frac{1}{375} = 5.0026$

2

7. Find the equation of the tangent line to the curve $y = 5x^2 - 3x^{15}$ at (1,2).

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$f'(x) = 10x - 45x^{14}$ 4

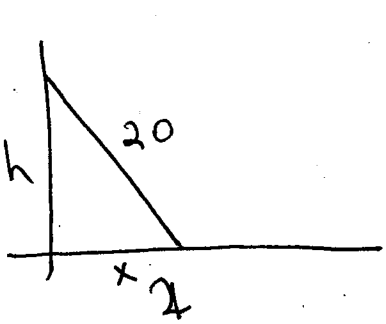
$f'(1) = 10 - 45 = -35$ 3

$y - 2 = -35(x - 1)$ $\to$ $y - 2 = -35x + 35$

$\boxed{y = -35x + 37}$

8. A ladder 20 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 2 ft/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 12 ft from the wall?

15



$\frac{dh}{dt} = ?$ when $x = 12$ and $\frac{dx}{dt} = 2$ ft/second

$20^2 = h^2 + x^2$ 2

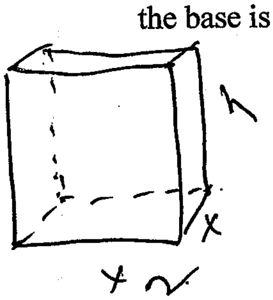
$2h\frac{dh}{dt} + 2x\frac{dx}{dt} = 0$ 2

$\frac{dh}{dt} = \frac{-x\frac{dx}{dt}}{h} = \frac{(-12)(2)}{\sqrt{20^2 - 12^2}}$ 3

$= \frac{-12(2)}{16} = -\frac{3}{2}$ ft/second

9. What are the dimensions of a box of maximum volume if the surface are is to be 18 square feet, the base is square, and the top is open?

15



$S = x^2 + 4xh = 18 \Rightarrow h = \frac{18 - x^2}{4x}$ 2

max Volume $= x^2 h = x^2\left(\frac{18 - x^2}{4x}\right) = \frac{x}{4}(18 - x^2) = \frac{1}{4}(18x - x^3)$ 2

$f(x) = \frac{1}{4}(18x - x^3)$

$f'(x) = \frac{1}{4}[18 - 3x^2]$

$3x^2 = 18$

2 $\quad x^2 = 6$

$x = \sqrt{6}$

$f'(\sqrt{6}) = 0$

$\to f''(x) = \frac{-6x}{4}$ 2

$f''(\sqrt{6}) < 0$

only critical value

f max when $x = \sqrt{6}$ 1

$h = \frac{18 - 6}{4\sqrt{6}} = \frac{3 \cdot 4}{4 \cdot}$

10. For $f(x) = x^4 - 8x^2 - 48$

a. Calculate the first and second derivative of $f(x)$.

6 (3) $f'(x) = 4x^3 - 16x$

(3) $f''(x) = 12x^2 - 16$

b. Find the intervals where $f(x)$ is increasing and decreasing.

7 $4x(x^2-4) = 4x(x-2)(x+2)$

$x+2$: $-$ $+$ $+$ $+$

$4x$: $-$ $-$ $+$ $+$

$x-2$: $-$ $-$ $-$ $+$

$(-)\ -2\ (+)\ 0\ (-)\ 2\ (+)$

2 f is inc when $x \in (-2,0) \cup (2,+\infty)$

2 f is dec when $x \in (-\infty,-2) \cup (0, -$

c. Find the intervals where $f(x)$ is concave up and concave down.

7 $12x^2 - 16 = 12(x^2 - \frac{4}{3}) = 12(x^2 - \frac{4}{3}) = 12(x + \frac{2}{\sqrt{3}})(x - \frac{2}{\sqrt{3}})$

$x + 2\sqrt{\frac{2}{3}}$: $-$ $+$ $+$

$x - 2\sqrt{\frac{2}{3}}$: $-$ $-$ $+$

$(+)\ -\frac{2}{\sqrt{3}}\ (-)\ \frac{2}{\sqrt{3}}\ (+)$

2 f is concave up on $(-\infty, -\frac{2}{\sqrt{3}}) \cup (\frac{2}{\sqrt{3}},$

2 f is concave down on $(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}$

d. Identify the local maximum, local minimum, and inflection points.

f has a local min at $x = \pm 2$ 2

max at $x = 0$ 2

6 inflection points at $\pm \frac{2}{\sqrt{3}}$ 2

e. Find the x- and y-intercepts of $y = f(x)$.

$f(0) = -48$ 2

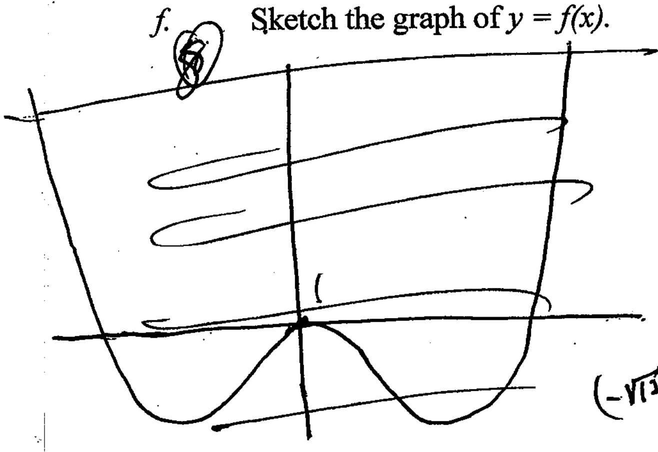
6 $(x^2-12)(x^2+4) = (x-\sqrt{12})(x+\sqrt{12})(x^2+4)$

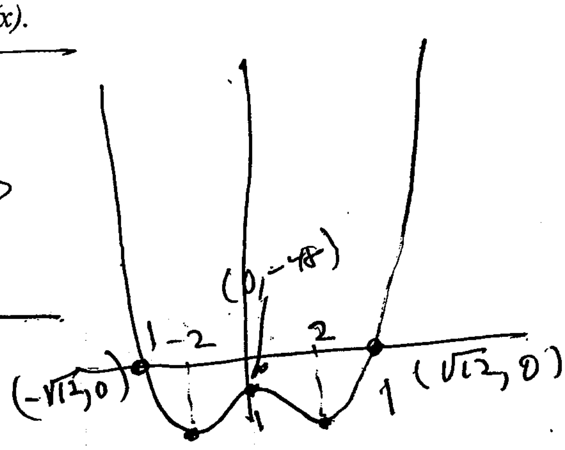
$f(\pm\sqrt{12}) = 0$

4

f. Sketch the graph of $y = f(x)$.

(6)





11. Calculate

a. $\int_2^4 \left(x^3 + \sin(\frac{\pi}{4}x)\right)dx$ $= \left(\frac{1}{4}x^4 - \cos(\frac{\pi}{4}x)\frac{4}{\pi}\right)\Big|_2^4$

$= \frac{1}{4}(4^4 - 2^4)\left[-\cos(\pi)\frac{4}{\pi} + \cos(\frac{\pi}{2})\frac{4}{\pi}\right]$

$= 4^3 - 4$ $60 + \frac{4}{\pi}$

61.2732

10

b. $\int \frac{1}{x(1+\ln x)^3}dx$ $= \int u^{-3}\,du = -\frac{1}{2}u^{-2} + C$ 3

3

$= -\frac{1}{2}\frac{1}{(1+\ln x)^2} + C$

1

10

$u = 1 + \ln x$ 1

$\frac{du}{dx} = \frac{1}{x}$ 1

$du = \frac{1}{x}dx$ 1

12. Use Newton's Method to find the next guess for the solution of $x^3 = 9x + 1$ if the first guess is $x_0 = 3$

10

Act. Answer 3.054842 2

$f(x) = x^3 - 9x - 1$

$f'(x) = 3x^2 - 9$

$x - \frac{f(x)}{f'(x)} = 3 - \frac{-1}{18} = 3 + \frac{1}{18} = 3.0\overline{5}$

2, 2

13. Find the area from $x = 1$ to $x = 3$ between the x-axis and the curve $y = 6x^2 + x$.

10

$\int_1^3 (6x^2 + x)\,dx = 2x^3 + \frac{1}{2}x^2 \Big|_1^3$

$= 2(27) + \frac{1}{2}(9) - (2 - \frac{1}{2})$

$= 54 + \frac{9}{2} - 2 - \frac{1}{2} = 52 + 4 = 56$

14. Evaluate the Riemann Sum for $f(x) = \frac{10}{x} + 2$, $5 \le x \le 13$, with four subintervals, taking the sample points to be midpoints.

10

5 7 9 11 13

$2(f(6) + f(8) + f(10) + f(12)) = 25.5$

$\Delta x = \frac{13-5}{4} = \frac{8}{4} = 2$

15. Find $f(x)$ if $f''(x) = x^2 + 2x$ and $f(0) = 2$ and $f'(0) = 1$.

$f'(x) = \frac{1}{3}x^3 + x^2 + k_1$ 3

$1 = f'(0) = 0 + 0 + k_1$ 1

$f'(x) = \frac{1}{3}x^3 + x^2 + 1$

$f(x) = \frac{1}{12}x^4 + \frac{1}{3}x^3 + x + k_2$ 3

$2 = f(0) = 0 + 0 + 0 + k_2$ 1

$f(x) = \frac{1}{12}x^4 + \frac{1}{3}x^3 + x + 2$ 1

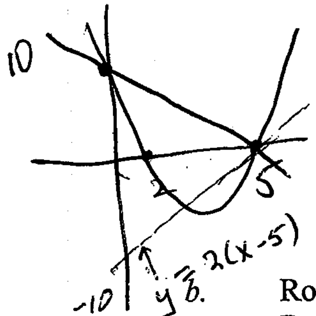
16. Find the average value of $f(x) = x + x^3$ on $[0,2]$. 3

$\frac{1}{2-0} \int_0^2 (x + x^3)\,dx = \frac{1}{2}\left(\frac{1}{2}x^2 + \frac{1}{4}x^4\right)\Big|_0^2$

$= \frac{1}{2}\left(\frac{1}{2}2^2 + \frac{1}{4}2^4\right)$ 1

$= \frac{1}{2}(2 + 4) = 3$ 1

17. a. Calculate the area between the curves $y = (x-2)(x-5)$ and $y = 2(5 - x)$.



$(x-2)(x-5) = 2(5-x)$

$(x-2)(x-5) + 2(x-5) = 0$

$(x-2+2)(x-5) = 0$

$x(x-5) = 0$

$\int_0^5 x(5-x)\,dx = \frac{5}{2}x^2 - \frac{1}{3}x^3\Big|_0^5$

$= \frac{5^3}{2} - \frac{5^3}{3} = \frac{1}{6}5^3$

$= \frac{125}{6}$

b. Rotate the area in (a) about the y-axis and give the integral that determines the value. Do not evaluate the integral.

10 $\int_0^5 2\pi x\left(x(5-x)\right)dx$

c. Rotate the area in (a) about the x-axis and give the integral that determines the value. Do not evaluate the integral.

$(x-2)(x-5) = 2(x-5)$

$(x-2)(x-5) - 2(x-5) = 0$

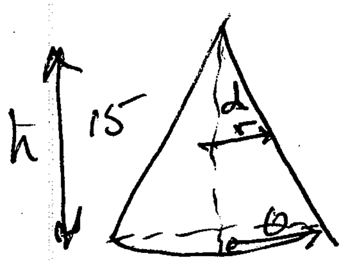
$(x-5)(x-2-2) = 0$

$(x-5)(x-4) = 0$

Answer is

$\int_0^4 \pi(2(5-x))^2\,dx + \int_4^5 \pi\left((x-2)(x-5)\right)^2 dx$

10 *Extra Credit: A cone with a base of radius 6 feet and height 15 feet is filled with water (62.5 lbs/cubic foot). How much work is required to pump the water out the top of the cone?*



$$\frac{d}{r} = \frac{15}{6}$$

$$r = \frac{6}{15} d$$

$$A(d) = \pi r^2 = \left(\frac{6}{15} d\right)^2 = \pi(.4d)^2 = \pi .16 d^2$$

$$\sum d_i \, 62.5 \, A(d_i) \, \Delta d \rightarrow \int_0^{15} \pi \, 62.5 \, .16 d^2 \, d \, D(d)$$

$$\int_0^{15} \pi (62.5)(.16) d^3 \, D(d)$$

$$\pi (62.5)(.16) \frac{d^4}{4} \Big|_0^{15}$$

$$\pi (62.5)(.16) \frac{15^4}{4}$$

$$\pi (2.5)(15^4) = 126\,562.5 \text{ ft-lbs.}$$