$z = x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}$ where $e^{i\theta} = \cos\theta + i\sin\theta$

$x = \text{Re}\, z$, $y = \text{Im}\, y$, $i^2 = -1$ $\qquad |z| = r = \sqrt{x^2+y^2}$ $\quad \tan\theta = \frac{y}{x}$

$\bar{z} = x - iy$, $z\bar{z} = |z|^2$ $\qquad \theta = \arg z$, $\arg z = \text{Arg}\, z + 2n\pi$

where $-\pi < \text{Arg}\, z \le \pi$

$\big||z_1| - |z_2|\big| \le |z_1 + z_2| \le |z_1| + |z_2|$

$-|z| \le \text{Re}\, z, \text{Im}\, z \le |z|$ $\qquad (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$$z^n = r_0 e^{i\theta_0} = r_0 e^{i(\theta_0 + 2k\pi)} \Rightarrow z = r_0^{\frac{1}{n}} e^{i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)}$$

Def: $\lim_{z \to z_0} f(z) = w_0$: Given $\epsilon > 0$ $\exists \delta > 0 \ni 0 < |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \epsilon$.

Def: $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$: f is differentiable at z_0

Thm: (Page 65): $f = u + iv$, diff at $z_0 \Rightarrow u_x = v_y$ and $u_y = -v_x$ at z_0
(Cauchy Riemann) and $f'(z_0) = u_x + iv_x$

Thm (Page 66): $f = u + iv$ defined in a ~~nbhd~~ neighborhood of z_0 along with u_x, v_x, u_y, v_y ; u_x, v_x, u_y, v_y cont at z_0 and CR at z_0, then $f'(z_0)$ exists and $f'(z_0) = u_x + iv_x$

(In Polar Coordinates $ru_r = v_\theta$ & $u_\theta = -rv_r$
$f'(z_0) = e^{-i\theta}(u_r + iv_r)$)

Def (Page 73) : f is analytic at z_0 if $f'(z)$ exists at each point of a nghd of z_0.

Def: entire function : analytic everywhere.

Def: $u(x,y)$ harmonic in a domain if cont. first & second order partials exist and $u_{xx} + u_{yy} = 0$ in the domain

Thm (Page 79): $f = u + iv$ analytic in $D \Rightarrow u, v$ harmonic in D

Def: (Page 80) harmonic conjugates : u, v harmonic in D & satisfy CR equations

$e^z = e^x e^{iy} = \sum_{k=0}^{\infty} \frac{1}{k!} z^k$ $\qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$; $\cos z = \frac{e^{iz} + e^{-iz}}{2}$

$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ $\qquad = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$

$\log z = \ln|z| + i(\arg z + 2n\pi)$

$\text{Log}\, z = \ln|z| + i(\text{Arg}\, z)$

(Be able to derive power series for $\log(1+z)$ using $\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$ and $\frac{d\log(1+z)}{dz} = \frac{1}{1+z}$)

Note: f analytic at $z_0 \Rightarrow$
$f(z) = \sum_{n=0}^{\infty} \frac{f^{n}(z_0)}{n!} (z - z_0)^n$
in a nghd of z_0.

Page 127: $\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$ Page 138: $\left|\int_C f(z)dz\right| \le ML$

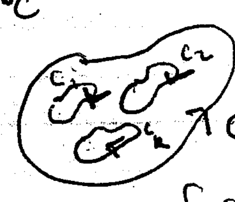
Page 142 f cont in D: TFAE (a) f has an antiderivative
(b) integration is path independent
(c) C ~~is~~ closed, $\int_C f(z)dz = 0$.

Page 151 Cauchy-Goursat thm: f analytic on and ~~inside~~ inside a simple closed ~~curve~~ contour C, then $\int_C f(z)dz = 0$.

Page 156 Def: Simply Connected domain D: ~~every simply~~ every simple closed contour in D encloses only points in D

Page 157: If f analytic in a simply connected domain, then $\int_C f(z)dz = 0$ for every closed contour in C.

Page 158-159



C simple closed ~~contour~~ contour (counterclockwise)
$C_1, \dots, C_k$ simple closed contours (clockwise)
interior to C, disjoint, no common interiors
f analytic on $C, C_1, \dots, C_k$
and interior to C $\cap$ exterior $C_1 \cap \dots \cap$ exterior C_k.

$\therefore \int_C f(z)dz + \sum_{n=1}^{k} \int_{C_n} f(z)dz = 0$.

$\therefore \int_C f(z)dz = \sum_{n=1}^{k} \int_{-C_k} f(z)dz$

Pages 164 - 165 - 166 Cauchy Integral Formulas:

$$\left.\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz \\ f'(z) &= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}dz \end{aligned}\right\} \Rightarrow f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

Page 170 $|f^{(n)}(z_0)| \le \frac{n!\, M_R}{R^n}$ Cauchy's Inequalities for the Derivatives

Immediate Consequences of these theorems are
Morera's thm (Page 169)
Liouville's Thm (Page 173)
Fundamental Thm of Algebra (Page 173)
Maximum Modulus Principle Page 175

$\sum_{k=0}^{\infty} z_k$ converges to z if $s_n = \sum_{k=0}^{n} z_k \to z$ as $n \to +\infty$.

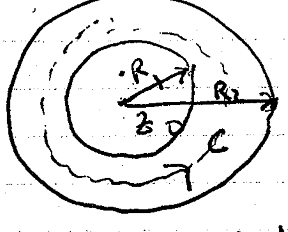
Review real convergence thms from calculus

Know absolute convergence $\Rightarrow$ convergence of series (page 186)

$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ if $|z|<1$; divergence otherwise

Taylor's Thm: (Page 189) f analytic in $|z-z_0|<R$, then $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$

Laurent's Series:



f analytic in $R_1 < |z-z_0| < R_2$, C in

then $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$ where $c_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$

$(n = 0, \pm 1, \pm 2, \pm 3, \ldots)$

(See Examples in ~~...~~ pages 202-205)

Review statements about absolute + uniform convergence of power series, continuity of power series, term-by-term differentiation + integration of power series, analyticity of power series, uniqueness of power series [Sections 63-66]. Multiplication + Division of Power Series [Section 67]

Def isolated singular point (page 229)

residue (page 231)

pole of order m page 241

removable singularity Page 242

essential singularity Page 243

Thm: Page 235 Cauchy's Residue Theorem

$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z)$

---

thm (Residues at Poles): $f(z)$ has pole of order m at z_0 iff $f(z) = \frac{\phi(z)}{(z-z_0)^m}$

where $\phi(z)$ is analytic and nonzero at z_0.

$\operatorname{Res}_{z=z_0} f(z) = \phi(z_0)$ if $m=1$

$\operatorname{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$ if $m \geq 2$

or $\lim_{z\to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]$

or if $f(z) = \frac{g(z)}{(z-z_0)^m \phi(z)}$ $g(z_0) \neq 0$, $\phi(z_0) \neq 0$

residue of f at z_0 = $\frac{d^{m-1}}{dz^{m-1}} \frac{g(z)}{\phi(z)}\Big|_{z=z_0}$

Section 85 $F(z) = \frac{P(z)}{q(z)}$

$\int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta = \int_{|z|=1} F\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)\frac{dz}{iz}$

Thm: $F(z) = \frac{P(z)}{g(z)}$ (i) $g(z) \neq 0$ on real axis

(ii) degree of $g \geq$ degree of P + 2

Sections 78, 79, 80, 81

then for $a \geq 0$ $\int_{-\infty}^{\infty} F(x) \begin{matrix}\cos ax\\ \sin ax\end{matrix}\,dx = \begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix} 2\pi i \sum_{\mathrm{Im}\, z > 0} \operatorname{Res} F(z) e^{iaz}$