In the below derivation, the x-coordinate is aligned with the direction of the undisturbed string, with y-coordinate the transverse direction aligned with the perturbation.

Overview: We seek a differential equation describing the vertical motion of the string, of the form $\ddot{y} = F(y)/m$ with F being the net restoring force and m a measure of the string's mass. An important first approximation is that the disturbance is of small amplitude, which is satisfied if the aspect ratio of the perturbation (wave) is small, *i.e.,* $A/L \ll 1$ (Figure 1a). We also assume stretching and horizontal motions are negligible, so the mass-per-unit-length is constant. We will begin by choosing an arbitrary segment Δx within the disturbed portion of the string and determine the net forces and acceleration of that segment (Figure 1b). Then take the limit $\Delta x \to 0$ (*e.g.,* , Figure 1c) to form a differential equation.

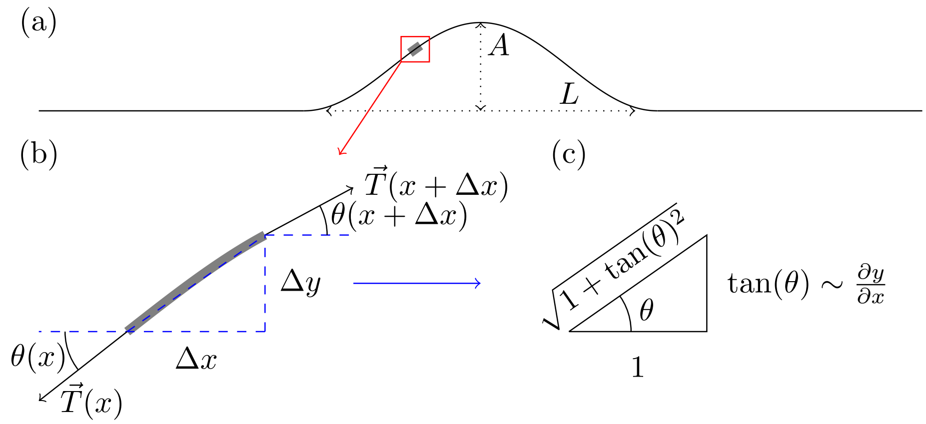

FIGURE 1. (a) Initial disturbance having an amplitude A over a length L on a taught string with initial tension T_0. (b) Zoomed in segment Δx illustrating the forces (tension vectors, $\vec{T}$) acting on the ends of the segment. (c) Unit right-triangle formed by joining the end-points of the segment in the limit $\Delta x \to 0$.

Set-Up: Newton's second law for the segment, in component form, is:

$$\hat{x}: \quad -T(x)\cos[\theta(x)] + T(x+\Delta x)\cos[\theta(x+\Delta x] = 0, \tag{1}$$

$$\hat{y}: \quad -T(x)\sin[\theta(x)] + T(x+\Delta x)\sin[\theta(x+\Delta x] = \mu\sqrt{\Delta x^2 + \Delta y^2}\ \ddot{y}, \tag{2}$$

where μ is the mass-per-unit-length of the string. Applying the definition of the derivative,

$$\frac{\partial F(x)}{\partial x} = \lim_{x\to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x},$$

to equations (1-2) gives,

$$\hat{x}: \quad \frac{\partial}{\partial x}\left[T(x)\cos\left(\theta(x)\right)\right] = 0, \tag{3}$$

$$\hat{y}: \quad \frac{\partial}{\partial x}\left[T(x)\sin\left(\theta(x)\right)\right] = \mu\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}\ \ddot{y}. \tag{4}$$

Evaluating the horizontal $\hat{x}$ derivative in (3) gives,

$$\frac{\partial T}{\partial x} = T\,\tan(\theta)\frac{\partial \theta}{\partial x}, \tag{5}$$

indicating that the tension varies in x depending on the string-slope (Figure 1c),

$$\tan(\theta) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{\partial y}{\partial x}. \tag{6}$$

and the string-curvature through $\partial\theta/\partial x$, *i.e.,* differentiating both sides of (6) with respect to x gives,

$$\begin{aligned}
\frac{\partial \theta}{\partial x}\sec(\theta)^2 &= \frac{\partial^2 y}{\partial x^2},\\
\frac{\partial \theta}{\partial x}\left(1+\tan(\theta)^2\right) &= \frac{\partial^2 y}{\partial x^2},\\
\frac{\partial \theta}{\partial x} &= \frac{\partial^2 y}{\partial x^2}\left(1+\left(\frac{\partial y}{\partial x}\right)^2\right)^{-1}.
\end{aligned} \tag{7}$$

Approximation: Before continuing to solve (4), it is useful to consider (4–7) under our original assumption that $A/L \ll 1$. Provided the initial disturbance is smooth, the string-slope is of a similar order in magnitude as the aspect ratio of the disturbance, *i.e.,* $\partial y/\partial x \sim \mathcal{O}(A/L)$. As such, we may simplify (4) and (7) by dropping the terms $(\partial y/\partial x)^2 \lll 1$, as they are *very small*! Thus, tension varies in x as, $T' = T\,y'y''$, where the prime $()'$ indicates derivatives w.r.t. x. This also simplifies our analysis of the vertical $\hat{y}$ component; differentiating and simplifying (4) using (6-7) and Figure 1c,

$$\begin{aligned}
T\cos(\theta)\theta' \quad &+ \quad T'\sin(\theta) &&= \mu\ddot{y},\\
Ty''\left(1+\cancelto{0}{(y')^2}\right)^{-3/2} &+ Ty''\cancelto{0}{(y')^2}\left(1+\cancelto{0}{(y')^2}\right)^{-1/2} &&= \mu\ddot{y},
\end{aligned}$$

gives,

$$\frac{\partial^2 y}{\partial t^2} - \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2} = 0, \tag{8}$$

the 1D linear wave equation with phase speed $v = \pm\sqrt{T/\mu}$.

Solution: There are several methods for solving (8), *e.g.,* change of variables, factorization (both discovered by d'Alembert), and separable plane-wave solution–but, these are beyond the scope of this blurb. The general form of the solution is,

$$y = \mathrm{A}\cos\left(\frac{2\pi}{\lambda_{\mathrm{A}}}(x - v\,t)\right) + \mathrm{B}\sin\left(\frac{2\pi}{\lambda_{\mathrm{B}}}(x - v\,t)\right), \tag{9}$$

where constants A, B and $\lambda_{(\mathrm{A,B})}$ are determined by initial and/or boundary conditions.

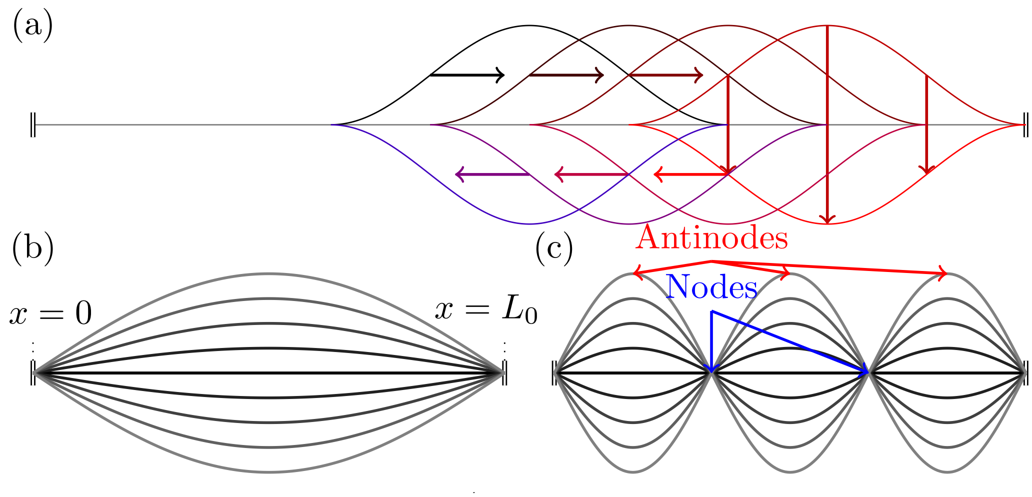

FIGURE 2. (a) Initial disturbance propagating to the right (black $\to$ red), switching polarity at the end of the string and returning (red $\to$ blue) shown at time intervals of $\Delta t \approx L/(4v)$. (b) Segment vibrating with fundamental wavelength $\lambda_0 = 2L_0$ shown at multiple phases of the oscillation. (c) Vibrations for mode $n = 2$ with wavelength $\lambda_2 = 2L_0/3$, at similar phases to (b).

Discussion: If the disturbance in Figure 1a propagates to the right with phase speed v and encounters a fixed endpoint, it will reverse direction and polarity (Figure 2a). The disturbance is forced to change polarity by the fixed end of the string. If confined to finite string of length L_0 (*e.g.,* fixed at $x = 0$ and $x = L_0$ Figure 2b) it will "bounce" back and forth from end to end. The fixed boundary conditions require the general solution (9) be modified. At time $t = 0$, we require $\mathrm{B} = 0$ and $\sin(2\pi L_0/\lambda_\mathrm{A}) = 0$, which is accomplished by forcing the argument to equal integer multiples of π, *i.e.,* $2\pi L_0/\lambda_\mathrm{A} \in \{\pi, 2\pi, 3\pi, ...\}$. Thus, only certain wavelengths called *modes* are allowable, with the n^{th} wavelength being $\lambda_n = 2L_0/(n+1)$ (Figure 2b-c). We must also require that the solution at $x = 0$ and $x = L_0$ be zero for all time. This is achieved by the superposition of two waves both with $\lambda = \lambda_n$ & $\mathrm{A} = \mathrm{A}_n$ propagating in opposite directions,

$$y_n(x,t) = \mathrm{A}_n \sin\left(\frac{2\pi}{\lambda_n}(x - vt)\right) + \mathrm{A}_n \sin\left(\frac{2\pi}{\lambda_n}(x + vt)\right), \qquad (10)$$

where the first term propagates to the right (wave features at (x,t) were initially at $(x - vt, 0)$) and the second term propagates to the left. Using the trigonometric identity, $\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \sin(\beta)\cos(\alpha)$, reduces the solution to,

$$y_n(x,t) = 2\mathrm{A}_n \sin\left(\frac{2\pi}{\lambda_n}x\right)\cos(\omega_n t), \qquad (11)$$

with $\omega_n = 2\pi v/\lambda_n$, and thus satisfying the boundary conditions for all time. Each modal solution (11) consists of a standing waveform with n zero amplitude interior nodes and $n+1$ large, time varying amplitude antinodes. The antinodal oscillation frequency f_n (units of cycles per second, Hz) increases linearly with increasing mode number. Based on the kinematic relation between speed, distance, and time, $v = \lambda_n f_n$, the frequency increases as $f_n = (n+1)\, v/L_0$. Therefore, the lowest possible frequency of oscillation is the fundamental mode f_0 and depends entirely on $v = \sqrt{T/\mu}$. Resonance occurs when external forcing has a frequency that matches f_n, causing the amplitude A_n to increase and become phase locked to the forcing.