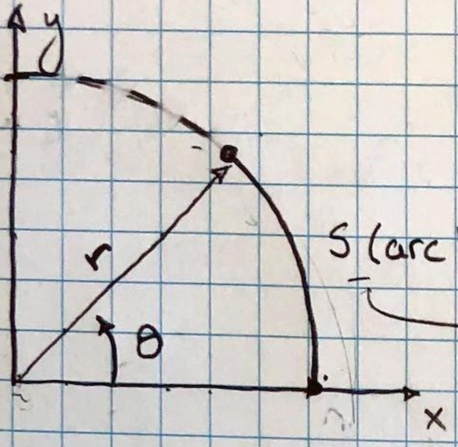
CH7: Rotational Motion



s (arc length) $= r\theta$ → Circumference $= \overset{\theta \text{ of full circle}}{2\pi}\, r$

$$\boxed{\theta = \frac{s}{r}}$$

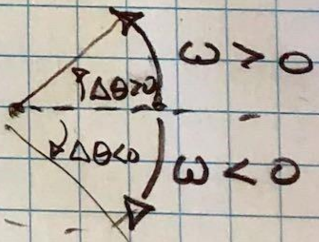
radians (rad): 2π (rad)

$2\pi = 360° \Rightarrow \boxed{\frac{\pi}{180°} = 1}$

$1 \text{ rad} \cdot \frac{180°}{\pi} = 57.3°$

Angular velocity:

$$\boxed{\omega \text{ (omega)} = \frac{\Delta\theta}{\Delta t} = \frac{\theta_f - \theta_i}{\Delta t}}$$



Recall:

$\Delta s = r\Delta\theta$

$$|v| = \frac{\text{dist}}{\text{time}} = \frac{\Delta s}{\Delta t} = r\frac{\Delta\theta}{\Delta t}$$

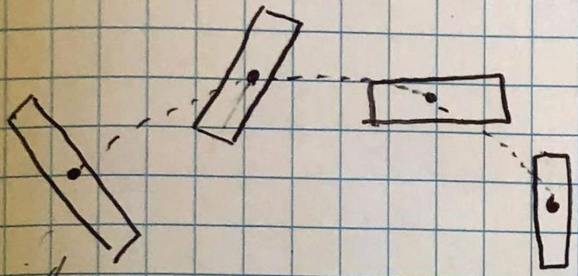
$$\therefore \boxed{\omega = \frac{v}{r}} \text{ or } \boxed{v = r\omega}$$

Also:

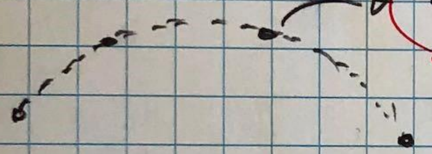
since, $v = \frac{2\pi r}{\tau}$ $\frac{\text{(circumference)}}{\text{(period)}}$

$$\boxed{\omega = \frac{2\pi}{\tau} = 2\pi f}$$

Rigid Body: shape + size do not change during motion!

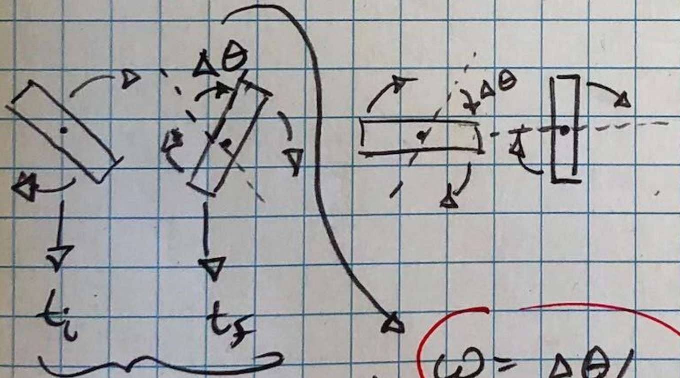


Combination of translation:



$$\vec{v} = \left(\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}\right)$$

And Rotation:



$$\omega = \Delta\theta / \Delta t$$

Angular Acceleration:

$$\boxed{\alpha\,(\text{alpha}) = \frac{\Delta\omega}{\Delta t}} = \frac{\overset{\text{final angular velocity}}{\omega_f} - \overset{\text{initial angular velocity}}{\omega_i}}{t_f - t_i}$$

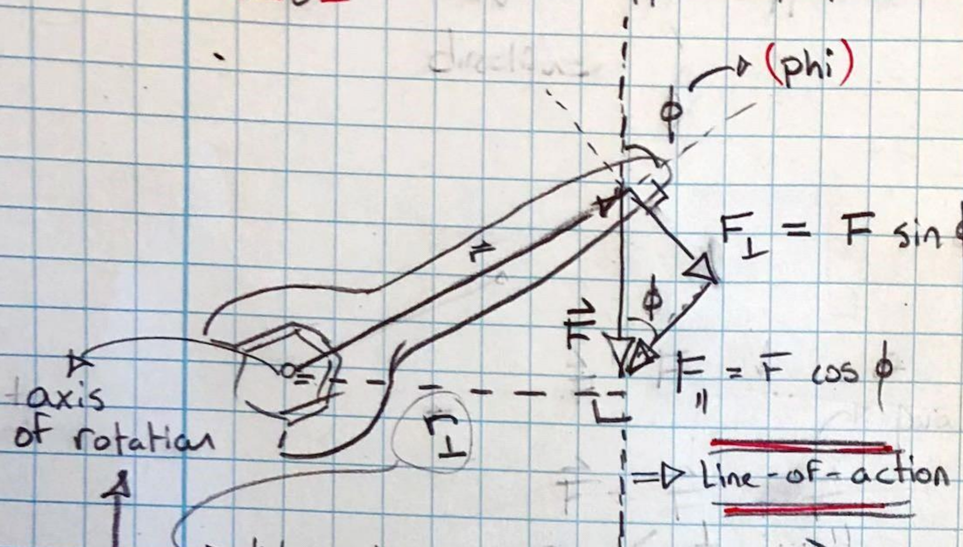
Tangential Acceleration:

$$a_t = \frac{\Delta V}{\Delta t} \quad \frac{(\text{change in speed})}{(\text{time})}$$

$$\boxed{\Delta V = \Delta\omega \cdot r}$$

$$\therefore a_t = \alpha \cdot r$$

Torque: Force applied perpendicular to the radius from a point.



$F_\perp = F\sin\phi$

$F_\parallel = F\cos\phi$

$\rightarrow \vec F = F_\perp + F_\parallel$

$F_\perp$: perpendicular to $\vec r$

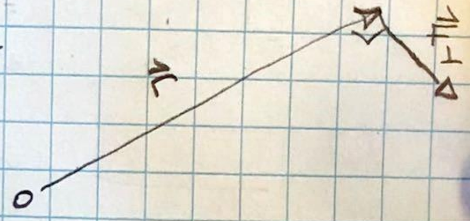
$F_\parallel$: parallel to $\vec r$

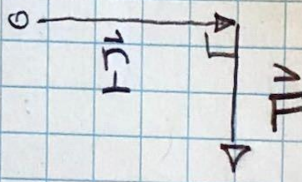
$r_\perp$: distance to center perpendicular to $\vec F$

$\vec r = \vec r_\perp + \vec r_\parallel$

$\vec r_\parallel$: parallel to $\vec F$

$$\boxed{\begin{aligned}\tau &= \pm r F_\perp \\ &= \pm r_\perp F\end{aligned}}$$





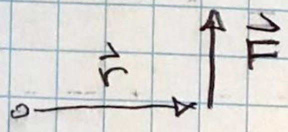
$= rF\sin\phi$

ϕ: angle between $\vec r$ + line of action ($\vec F$)

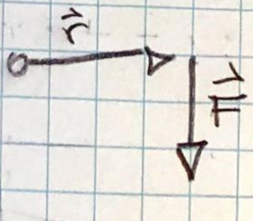
$r_\perp$ = Moment Arm

τ = torque about axis

$\tau > 0 \Rightarrow$

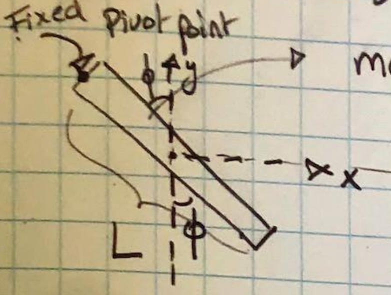


$\tau < 0 \Rightarrow$



$\tau_{net} = \tau_1 + \tau_2 + \dots$

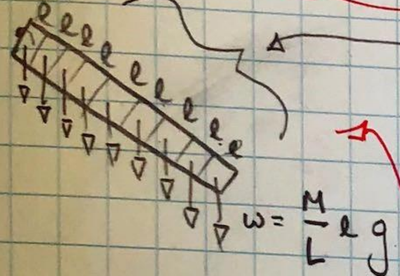
Gravitational torque + Center of Gravity (CoG)



mass $M = \frac{\text{mass}}{\text{Length}} \cdot L$

$\rightarrow L = l \cdot N$

Gravity pulls on each segment of length

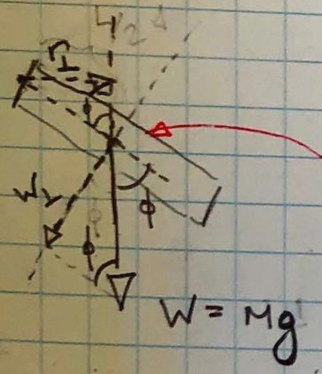


$$\tau = \left(\frac{M}{L} g\right) \sin\phi \left(l \cdot \frac{l}{2} + l\left(\frac{l}{2} + l\right) + l\left(\frac{l}{2} + 2l\right) + \dots\right)$$

$$= \frac{M}{L} g \sin\phi \, \frac{l^2}{2} \underbrace{\sum_{n=0}^{N-1} (1 + 2n)}_{N^2}$$

$$= \frac{M}{L} g \sin\phi \, \frac{(Nl)^2}{2}$$

$$= \frac{M}{L} g \sin\phi \, \frac{L^2}{2}$$

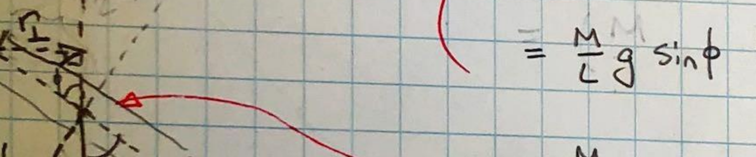


$$= M g \sin\phi \, \frac{L}{2} \rightarrow \text{center of Gravity!}$$

$$= W_\perp r = r_\perp W$$

Weight Force acts on Center of Gravity (CoG)!

If the pivot is located @ CoG then, $\tau = 0$!



$$\tau_0 = {}^{+}M_0 l_0 - M_1 l_1 - M_2 l_2 = 0$$

$$M_0 (X_{COG} - X_0) - M_1 (X_1 - X_{COG}) - M_2 (X_2 - X_{COG}) = 0$$

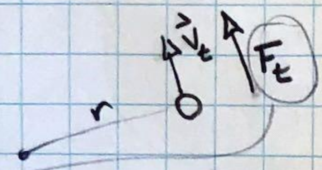
$$(M_0 + M_1 + M_2) X_{COG} = X_0 M_0 + X_1 M_1 + X_2 M_2$$

$$\left(X_{COG} = \frac{(X_0 M_0 + X_1 M_1 + X_2 M_2)}{(M_0 + M_1 + M_2)} = \sum_i m_i x_i \Big/ \sum_i m_i \right)$$

Torque causes Angular Acceleration:

$$\tau \longrightarrow \alpha$$

Recal: tangential acceleration: $a_t = \alpha \overset{\text{radius}}{r}$



tangential Force $F_t = m\,a_t$

$= m\,r\,\alpha$

$$\alpha = \frac{F_t}{mr}$$

$$\tau = F_\perp \cdot r = F_t\,r$$

$$\therefore \boxed{\alpha = \frac{\tau}{mr^2}} + \boxed{\tau = (mr^2)\,\alpha}$$

moment of Inertia (single particle)

Newton's 2nd Law (Rigid Body Rotational Motion)

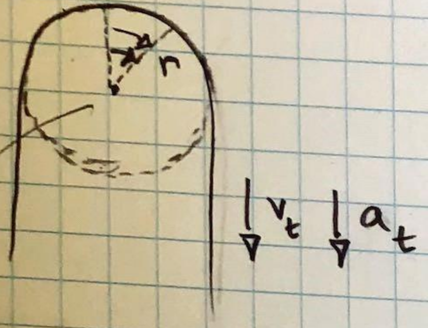
$$\tau_{net} = \tau_1 + \tau_2 + \ldots = \sum_i \tau_i = (m_1 r_1^2 + m_2 r_2^2 + \ldots)\alpha = \alpha \sum_i m_i r_i^2$$

Moment of Inertia $= \sum_i m_i r_i^2 = I$

$$\therefore \boxed{\tau_{net} = I\alpha}$$

rotational equivalent of Mass

Pulleys



$$v_t = \overset{\text{angular velocity}}{\omega_{pulley}}\, r$$

$$a_t = \underbrace{\overset{\text{angular acceleration}}{\alpha_{pulley}}}_{\frac{\tau}{I_{pulley}}}\, r$$

$$I_{pulley} = \overset{\text{mass of pulley}}{M_{pulley}}\, \frac{r^2}{2}$$

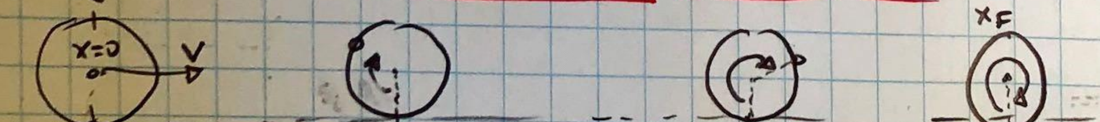
$$\therefore \boxed{a_t = \frac{2\tau}{(M_{pulley} \cdot r)}}$$

a_t **decreases** w/ M_{pulley} + r!

however, since $\tau = F_{\perp} \cdot r$

$$\boxed{a_t \sim \frac{F_{\perp}}{M_{pulley}}}$$

Rolling Motion: Bottom of wheel does not skid!

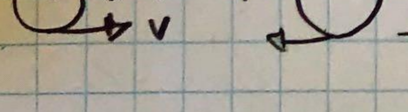


Point on ground @ t=0

point returns to ground when $t = \text{period} = \frac{2\pi r}{v}$

$$v = \frac{\Delta x}{\Delta t} = \frac{\text{Circumference}}{\text{Period}} = \underbrace{\frac{2\pi r}{\tau}}_{\omega} = \omega r$$

Combo of Rot + trans:



$$=$$

$$v + v_t = 2\omega r$$

$$v = \omega r$$

$$v - v_t = 0$$