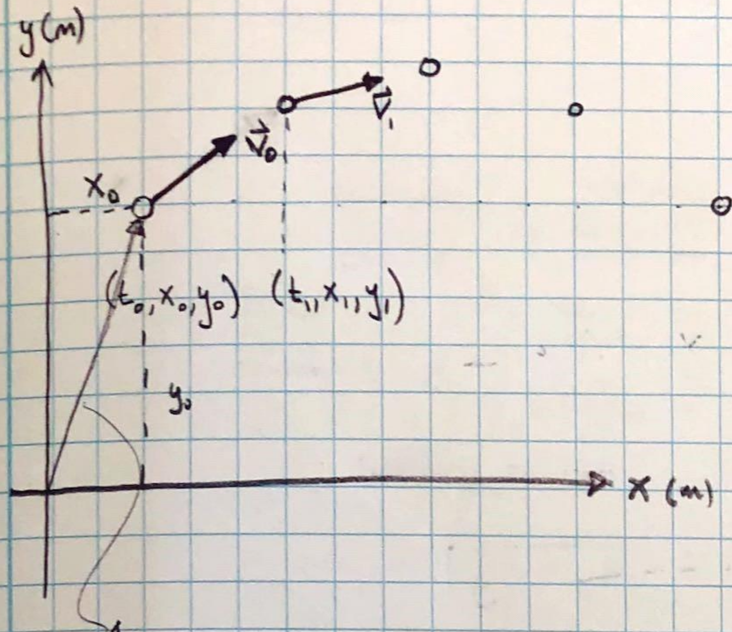
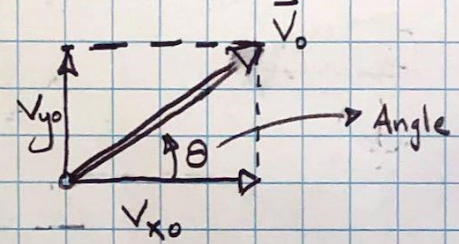
CH 3: Vectors & Motion in 2D



position vector $\vec{x}_0 = (x_0, y_0)$

velocity vector $\vec{V}_0 = (V_{x0}, V_{y0}) \Rightarrow$



"Velocity components"

Size $(\vec{V}_0)$

$|\vec{V}_0|$ (magnitude of $\vec{V}_0$) $= \sqrt{V_{x0}^2 + V_{y0}^2}$ → Pythagorean theorem (570 BC)

Direction $(\vec{V}_0)$

$\hat{V}_0$ (unit vector) $= \left(\frac{V_{x0}}{|\vec{V}_0|}, \frac{V_{y0}}{|\vec{V}_0|}\right) = \left(\frac{V_{x0}}{\sqrt{V_{x0}^2+V_{y0}^2}}, \frac{V_{y0}}{\sqrt{V_{x0}^2+V_{y0}^2}}\right)$

$|\hat{V}_0| = 1$

Angle $(\vec{V}_0) = \theta$

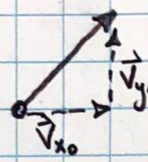
$\tan(\theta) = \left(\frac{V_{y0}}{V_{x0}}\right) \Rightarrow \theta = \tan^{-1}\left(\frac{V_{y0}}{V_{x0}}\right)$

"arc tangent" or "tangent inverse"

Vector Addition:

Say @ $t = t_0 = 0$,

$$\vec{v}_0 = $$



$$= \underbrace{\vec{V}_{x0}}_{\text{decomposed into Components}} + \vec{V}_{y0}$$

$\vec{V}_{x0}$ → x-component

$\vec{V}_{y0}$ → y-component

and that between $t_0 \to t_1$ there is constant acceleration due to gravity ($\vec{a} = a_y = -g$).

The change in velocity between $t_1 \to t_2$ is,

$$\Delta\vec{V} = \vec{a}\,\Delta t$$

Components

$$\Delta V_x = 0 \cdot \Delta t = 0$$

$$\Delta V_y = a_y\,\Delta t = -g\,\Delta t =$$

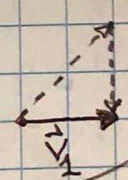
∴ (therefore) @ $t = t_1$,

$$\vec{V}_1 = \vec{V}_0 + \Delta\vec{V}$$

$$= \vec{V}_0 + \vec{a}\,\Delta t =$$

+

=



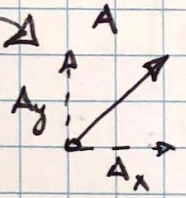
Components

$$V_{x1} = V_{x0} + 0 = V_{x0}$$

$$V_{y1} = V_{y0} + (-g\,\Delta t) = 0 \text{ (based on picture)}$$

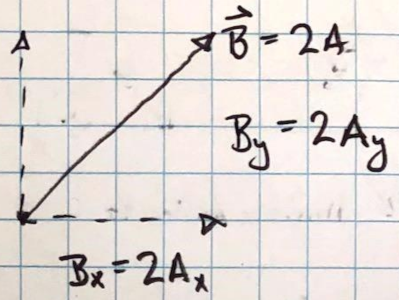
Scalar Multiplication:

$$\vec{B} = 2\vec{A}$$



$$(B_x, B_y) = 2(A_x, A_y)$$
$$= (2A_x, 2A_y)$$

or $2\vec{A} = \underbrace{\vec{A}+\vec{A}}_{\text{add components}}$

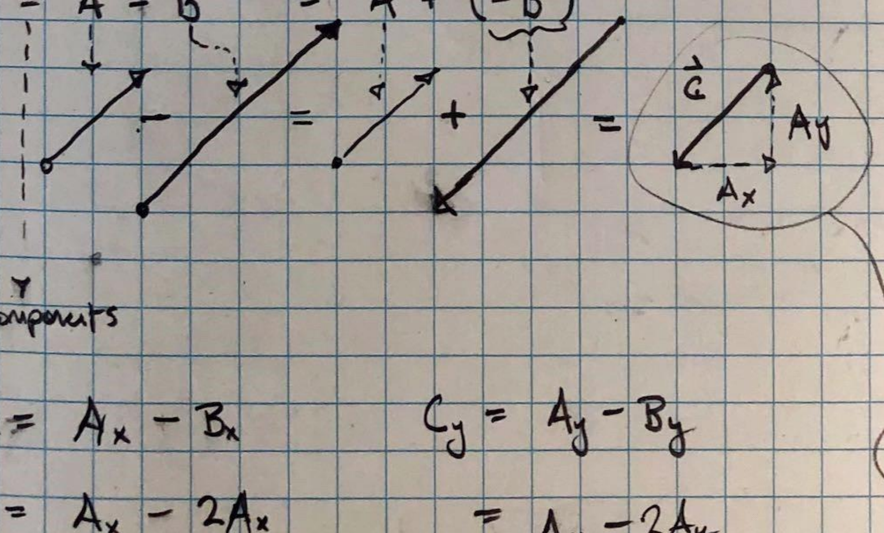


$$|\vec{B}| = |2\vec{A}|$$
$$= \sqrt{(2A_x)^2 + (2A_y)^2} = \sqrt{4(A_x^2 + A_y^2)} = 2\sqrt{A_x^2 + A_y^2}$$

$$\therefore \ |\vec{B}| = 2|\vec{A}|$$

Subtraction = Combo of multiplication (by −1) and addition

$$\vec{C} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$



components

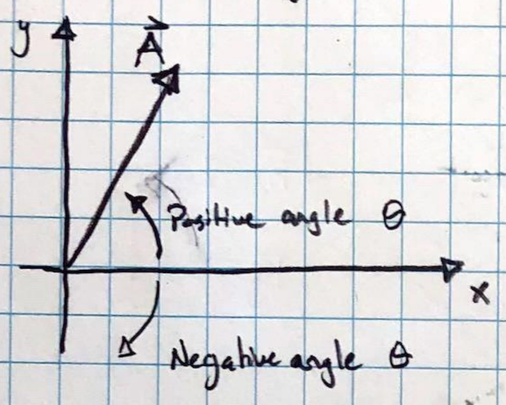
$$C_x = A_x - B_x \qquad C_y = A_y - B_y$$
$$= A_x - 2A_x \qquad = A_y - 2A_y$$
$$= -A_x \qquad = -A_y$$

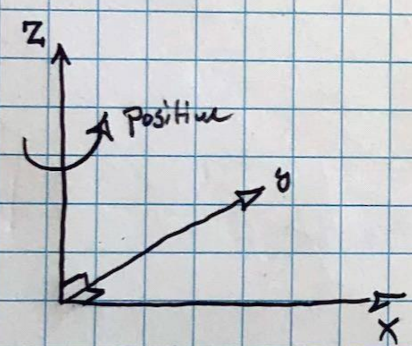
$$\vec{C} = -\vec{A}$$

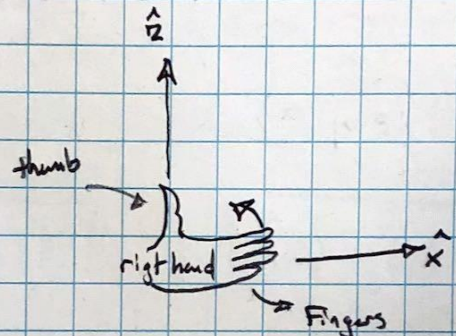
Cartesian Coordinates (right Handed)

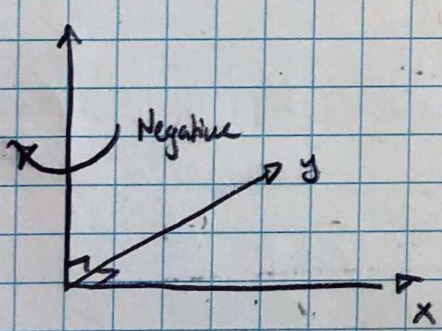
↓

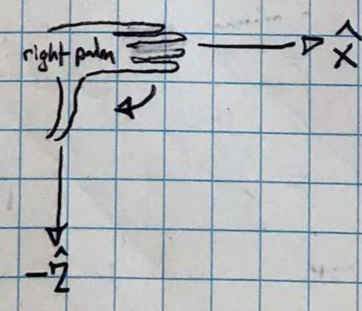
rectangular grid



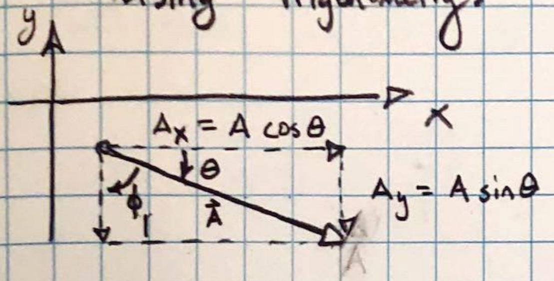








Using Trigonometry:



θ = theta

ϕ_1 = phi (pronounced Fee)

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\text{Hypotenuse} = |\vec{A}| = A = \sqrt{A_x^2 + A_y^2}$$

if:

$$\theta + \phi_1 = \boxed{-\frac{\pi}{2}\ (\text{radians})} = -90^\circ$$

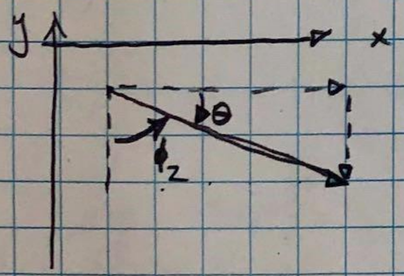
$$\phi_1 = -90 - \theta$$

$$\therefore \quad A_y = -A\cos(\phi_1)$$

$$A_x = -A\sin(\phi_1)$$

☆ Caution!!

if ϕ is measured from negative y-axis, i.e.,

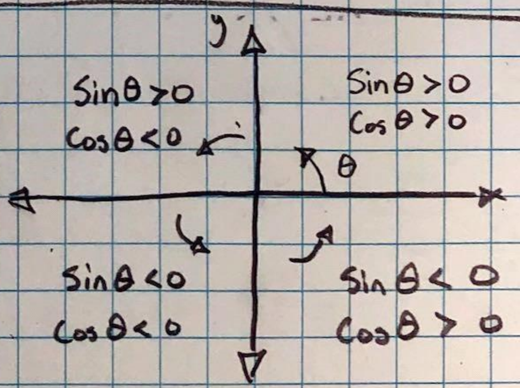


then $\theta - \phi_2 = -90^\circ$

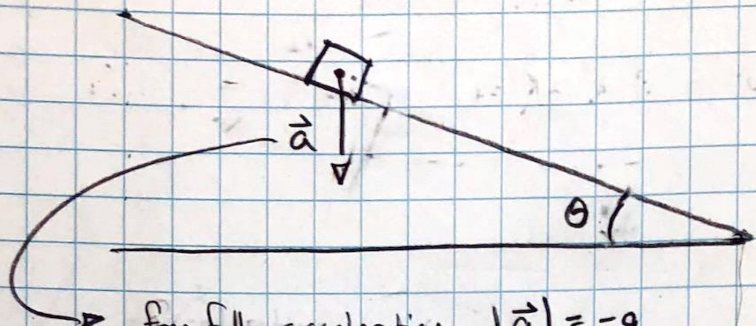
& $\phi_2 = 90^\circ + \theta$

$$\therefore \quad A_y = -A\cos(\phi_2)$$

$$A_x = A\sin(\phi_2)$$

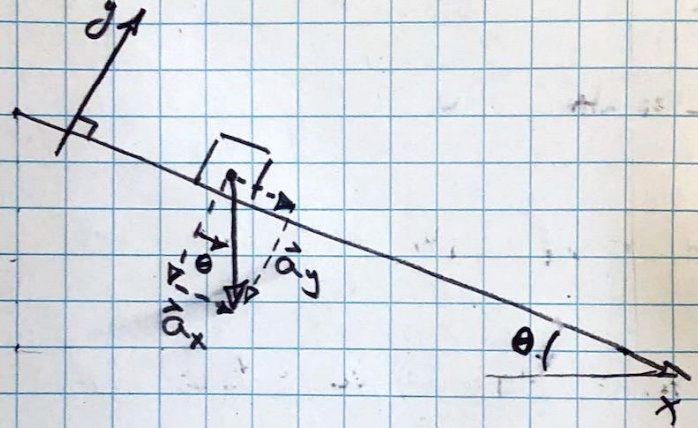


Block on inclined (Frictionless) plane



free fall acceleration $|\vec{a}| = -g$

define coords w/ +x down ramp & +y perpendicular to ramp



$$|\vec{a}_x| = |\vec{a}| \sin\theta$$

$$\therefore \quad \frac{\Delta \vec{V}}{\Delta t} = \vec{a}$$

Components

$$\frac{\Delta V_x}{\Delta t} = \vec{a}_x$$

$$= |\vec{a}| \sin\theta$$

$$= g \sin\theta$$

$\frac{\Delta V_y}{\Delta t} = 0$ ... Because ramp supports weight of Block!

$\boxed{\text{if } \theta = 0 \text{ (Level ramp)} \quad \frac{\Delta V_x}{\Delta t} = 0 \text{ (no motion)}}$

$\boxed{\text{if } \theta = 90^\circ \text{ (vertical ramp)} \quad \frac{\Delta V_x}{\Delta t} = g \text{ (Free fall)}}$

mental check!

For Multi-dimensional Motion,

define displacement vector $\Delta\vec{x} = (\Delta x, \Delta y, \Delta z) = \vec{d} = (d_x, d_y, d_z)$

then velocity vector is,

$$\vec{v} = \frac{\Delta\vec{x}}{\Delta t} = \frac{(\Delta x, \Delta y, \Delta z)}{\Delta t} = \frac{\vec{d}}{\Delta t}$$

Such that the acceleration vector is

$$\vec{a} = \frac{\Delta\vec{v}}{\Delta t}$$ as before, with

$$\vec{a}_x = \frac{\Delta\vec{v}_x}{\Delta t} \quad ; \quad \vec{a}_y = \frac{\Delta\vec{v}_y}{\Delta t} \quad ; \quad \vec{a}_z = \frac{\Delta\vec{v}_z}{\Delta t}$$

$\therefore$ the i^{th} component velocity + position are (For constant acc a_i)

$$v_{if} = v_{ii} + a_i \Delta t$$

$$d_{if} = d_{ii} + v_{ii}\Delta t + \frac{1}{2} a_i \Delta t^2$$

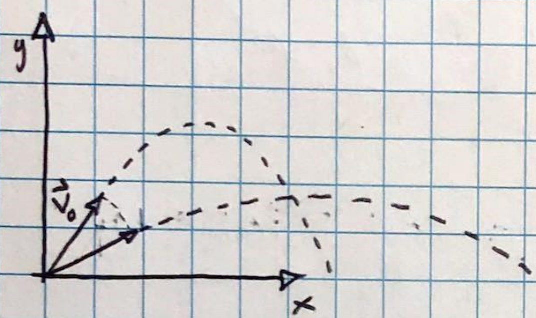
e.g.,
$$v_{xf} = v_{xi} + a_x \Delta t$$

$$d_{xf} = d_{xi} + v_{xi}\Delta t + \frac{1}{2} a_x \Delta t^2$$

Projectile motion (neglecting drag)

$$\vec{a} = -g \downarrow \quad \text{(points to Earth's center... roughly)}$$

trajectory (path) depends on initial velocity $\vec{V_i}$



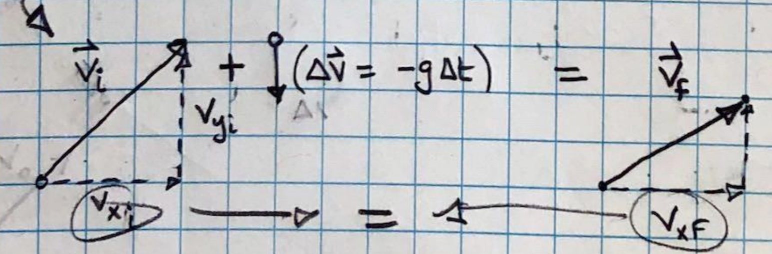
Since $\vec{a} = (\vec{a}_x, \vec{a}_y) = (0, -g)$

$V_{xf} = V_{x_i}$; bce, $\Delta V_x = a_x \Delta t = 0 \cdot \Delta t$

whereas,

$V_{yf} = V_{y_i} + a_y \Delta t = V_{y_i} - g\,\Delta t$

Geometrically



$$\boxed{V_{yF} = V_{y_i} - g\Delta t}$$

Position of Projectile: $\vec{d}_f = \vec{d}_i + \vec{v}_i \Delta t + \frac{1}{2}\vec{a}\,\Delta t^2$

$\Delta x = V_{xi}\Delta t$

$\Delta y = V_{yi}\Delta t - \frac{1}{2} g \Delta t^2$

if $\Delta y = 0$, i.e., y_i … y_F

then, $V_{yi}\Delta t - \frac{1}{2} g \Delta t^2 = 0$

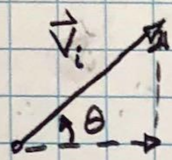
$$\boxed{\Delta t = \frac{2V_{yi}}{g}} \quad \& \quad \boxed{\Delta x = \frac{2V_{xi}V_{yi}}{g}}$$

Also $\boxed{V_{yF} = -V_{yi}}$

What angle gives maximum Δx ?

$$\Delta x = \frac{2 V_{xi} V_{yi}}{g}$$

Recall :



$V_{yi} = |\vec{V}_i| \sin\theta$

$V_{xi} = |\vec{V}_i| \cos\theta$

s.t. (such that)

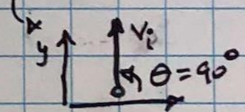
$$\Delta x = \frac{|\vec{V}_i|^2}{g} \cdot 2 \sin\theta \cos\theta$$

Trig. Identity

$$\boxed{2 \sin\theta \cos\theta = \sin(2\theta)}$$

$$\Delta x = \frac{|\vec{V}_i|^2}{g} \sin(2\theta)$$

$\to = 0$ when $\theta = 0$ or $90°$

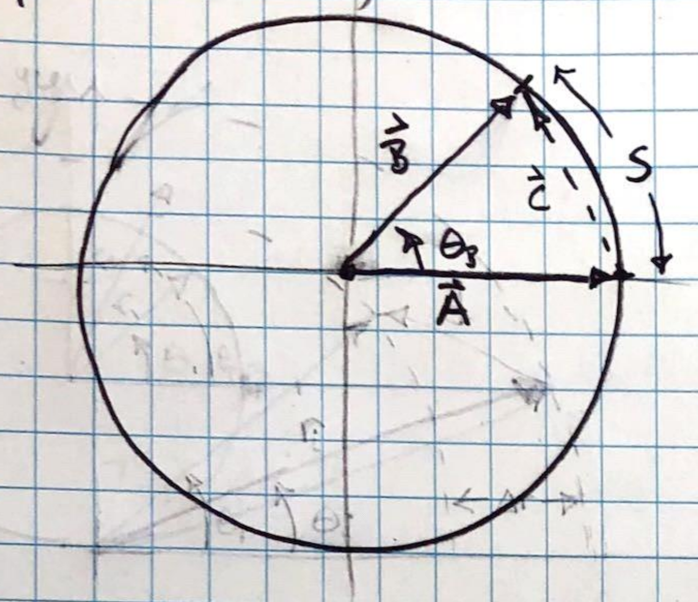


$\to \boxed{= 1 \text{ when } \theta = 45°}$

$$\boxed{\Delta x_{max} = \frac{|V_i|^2}{g} \qquad \theta_{max} = 45°}$$

Circular Motion

(polar coordinates) = radius + angle, or r & θ



$r = |B| = |A|$

$\theta_B = \tan^{-1}\left(\frac{B_y}{B_x}\right)$

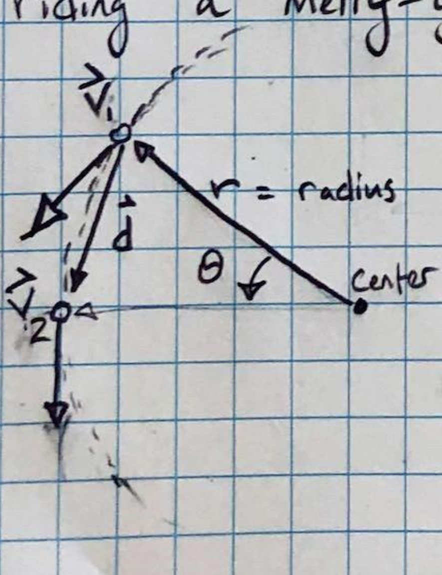
$|B| = |A|$; but $\theta_A = 0$ & $\theta_B \neq 0$ $\Rightarrow$ $\Delta\theta = \theta_B - \theta_A =$

$\vec{C} = \vec{B} - \vec{A} \approx \left(S = \underbrace{|A|}_{\text{radius}}\ \underbrace{\Delta\theta}_{\text{angle in radians}}\right)$ for small angles

"Small angle approximation"

Must use RADIAN ANGLES

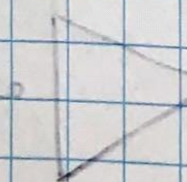
riding a merry-go-round (constant speed)



$|V_1| = |V_2| = \frac{S}{\Delta t} \approx \frac{\vec{d}}{\Delta t}$

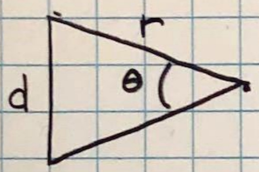
(Δt is small so θ is small)

$\Delta\vec{V} = \vec{V}_2 - \vec{V}_1$

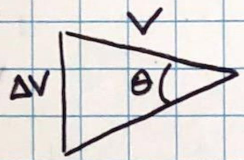


ΔV → points toward center of circle

The two equilateral triangles



and



are geometrically similar

$$\therefore \quad \boxed{\frac{d}{r} = \frac{\Delta V}{V}}$$

recall : $V = |V_1| = |V_2| = \dfrac{d}{\Delta t}$

s.t. : $\boxed{d = V \Delta t}$

combine

$$\left\{ \frac{V \Delta t}{r} = \frac{\Delta V}{V} \right\} \cdot \frac{V}{\Delta t}$$

$$\boxed{\frac{\Delta V}{\Delta t} = \frac{V^2}{r} = a_c}$$ (centrepital acceleration)

"center seeking"