Early Childhood Education Journal, Vol. 35, No. 1, August 2007 (© 2007)
DOI: 10.1007/s10643-007-0173-8

Peer Reviewed

Elementary School Students' Mental Computation Proficiencies

Filiz Varol[1,2] and Dale Farran[1]

Mental computation helps children understand how numbers work, how to make decisions about procedures, and how to create different strategies to solve math problems. Although researchers agree on the importance of mental computation skills, they debate how to help students develop these skills. The present study explored the existing literature in order to identify key points that are related to students' use of different mental calculation strategies in a variety of settings and their conceptual understanding of those strategies.

KEY WORDS: young children; mathematics; mental computation.

INTRODUCTION

The term number sense refers to a person's general understanding of numbers and operations along with the ability to use this understanding in flexible ways to manage numerical situations (McIntosh, Reys, & Reys, 1992). Although children develop number sense in informal ways in their early years, after the age of four or five, they start their school lives and learn numbers and operations in more formal ways (Varol & Farran, 2006). In the past, the primary mathematics computation in early school years was based on the pen-and-paper algorithm (Cooper, Heirdsfield, & Irons, 1996). However, today researchers have realized the importance of mental computation, and have been exploring its effects on students' success in mathematics and the factors that influence children's accuracy and flexibility. In order to address the need to improve mathematics instruction that leads elementary school children to be accurate and flexible in using a variety of strategies to solve math problems, the present study investigated the existing literature on elementary school students' multidigit mental computation strategies. In the first section, the definitions for computation and mental computation were given. The second section explored the mental computation strategies that are used by elementary school students. While the third section dealt with the association between conceptual understanding and procedural skills, the last section investigated the existing literature in order to address whether it is crucial to teach mental computation skills to elementary school students.

[1]Department of Teaching and Learning, Vanderbilt University, Peabody # 330 230 Appleton Place, Nashville, TN 37203-5721, USA.
[2]Correspondence should be directed to Filiz Varol, Department of Teaching and Learning, Vanderbilt University, Peabody # 330 230 Appleton Place, Nashville, TN 37203-5721, USA., e-mail: filiz.varol@vanderbilt.edu

COMPUTATION VERSUS MENTAL COMPUTATION

In the literature, written computation was defined as following: to manipulate numbers "on paper" to achieve the desired answer (Mclellan, 2001). As noted above, the primary mathematics computation in early school years was based on the pen-and-paper algorithm (Cooper et al., 1996). However, today, a growing body of studies focuses on mental calculation and tries to determine its influences on elementary school students' success in and out of school. Unlike the pen and paper algorithm, mental calculation is the process of carrying out arithmetical operations without using external devices including pen, paper, and calculators (Maclellan, 2001; Reys, 1984).

According to the studies (Cobb & Merkel, 1989; Klein & Beishuizen, 1994; Maclellan, 2001; Reys, 1984, 1985; Sowder, 1992), mental computation for addition and subtraction of multidigit numbers plays a significant role on teaching children how numbers work, how to make decisions about procedures, and how to create different strategies to solve math problems. According to Sowder, children who are unskilled with computers ‘‘ignored even obvious number properties that would help them and depended wholly on mentally reproducing the standard paper-and-pencil algorithms’’ (1992, p. 14). In other words, the research on mental computation proposes the connection between mental computation and conceptual understanding of numbers. Specifically, Maclellan (2001) points out the most important difference between calculation with external devices and mental calculation: whereas use of written algorithms encourages children to follow different steps without thinking about what they are doing, mental computation allows them to involve into the process to determine what the numbers in the problem mean. In today’s society, mental calculation is also required because, for instance, it is meaningless to ask for a calculator to compute the amount of tip we are planning to give a waitress. Since the studies given above emphasize the importance of mental computation, the following section aimed to explore mental computation strategies that elementary school students tend to use.

MENTAL COMPUTATION STRATEGIES

The literature included a variety of strategies for mental addition and subtraction for elementary school students (Blote, Klein, & Beishuizen, 2000; Cooper et al., 1996; Heirdsfield & Cooper, 2004a, 2004b; Thompson & Smith, 1999). These strategies are N10, N10C, 10s, 1010, A10, counting, short jump, and mental image of pen and paper algorithm (see Table 1). According to the literature, among these strategies, N10, 1010, and mental image of pen and paper algorithm are the strategies that are used widely by students.

Before discussing the differences among N10, 1010 and mental image of pen and paper algorithm, it is essential to define these strategies. In the N10 strategy the second number in a written expression of an addition or subtraction problem is split into units and tens that are subsequently added or subtracted. The second strategy, 1010, requires splitting both of the numbers in the written expression into units and tens for summing and subtracting separately and at the end the results are reassembled (Lucangeli, Tressoldi, Bendotti, Bonanomi, & Siegel, 2003). On the other hand, mental image of pen and paper algorithm can be described as employing strategies that reflect pen and paper algorithms mentally.

Studies showed that the 1010 strategy is favorably used in the United States because it decomposes numbers in tens and ones, which is taught from early grades of primary schools in the United States (Blote et al., 2000; Heirdsfield & Cooper, 2004b; Klein & Beishuizen, 1994). On the other hand, in European countries the N10 strategy is emphasized because the N10 strategy may minimize the percentage of errors students make (Blote et al., 2000; Heirdsfield & Cooper, 2004b; Klein & Beishuizen, 1994; Klein, Beishuizen & Treffers, 1998). For instance, while a problem like ‘36 + 27’ can be solved with the 1010 strategy, a problem like ‘74−69’ can be solved with N10 strategy. For the latter type of problem, there might be two disadvantages for students if they use the 1010 strategy: the workload on the memory and the chance that errors occur. Blote et al. (2000) conducted a study with Dutch second grade students in order determine which mental strategies students preferred to use while solving addition and subtraction problems. In this study, participants were introduced the mental strategies, N10 and 1010, separately. They found that the N10 strategy was

Table 1. Mental Strategies for Addition and Subtraction

Strategies	Addition Examples 36 + 27	Subtraction Examples 74−69
N10	36 + 20 = 56; 56 + 7 = 63	74−60 = 14; 14−9 = 5
N10C	36 + 30 = 66; 66−3 = 63	74−70 = 4; 4 + 1 = 5
10s	30 + 20 = 50; 50 + 6 = 56; 56 + 7 = 63	70−60 = 10; 10 + 4 = 14; 14−9 = 5
1010	30 + 20 = 50; 6 + 7 = 13; 50 + 13 = 63	70−60 = 10; 4−9 = −5; 10 + (−5) = 5
A10	36 + 4 = 40; 40 + 23 = 63	74−4 = 70; 70−65 = 5
Counting		Counting backward from 74 to 69
Short jump		69∩70∩74 = 1 + 4 = 5
Mental image of pen and paper algorithm	Using the pen and paper algorithm mentally	

dominant procedure used by second graders for both addition and subtraction problems.

As noted above, in addition to N10 and 1010 strategies, mental image of pen and paper algorithm is also used by students. However, this strategy is used by inflexible students and it is considered as an inefficient strategy (Heirdsfield & Cooper, 2004a; Sowder, 1992). Maclellan describes the mental image of pen-and-paper algorithm as an algorithm that "treats numbers as single digits and adopt a uniform approach to executing, say, subtraction operations" (2001, p. 146) Heirdsfield and Cooper (2004a) interviewed with third grade students in order to determine their understanding of mental computation, number facts, computational estimation, numeration, and effect of operation on number. They reported that only inflexible children mentioned that they saw the numbers in their head when calculating. According to the authors, "they imagined the numbers moving, one under the other, to represent the algorithm" (2004a, p. 455). Although conceptual understanding and its association with one's mental computation skills have been addressed in the literature, it might be valuable to explore how conceptual understanding and computation skills develop.

CONCEPTUAL UNDERSTANDING AND PROCEDURAL SKILLS

In the present paper, conceptual understanding was borrowed from the existing literature that considered understanding as the internal construction of connections or relations among representations of mathematical ideas (Hiebert & Wearne, 1996). As an example, children who have conceptual understanding recognize that multidigit numbers consist of ones, tens, hundreds and so on by referring to the idea of decomposing. Although the relationship between conceptual understanding and procedural skills has been the target of many studies, there is little empirical information on how mathematical understanding and skills interact and develop together during the instruction time. For instance, according to Hiebert and Wearne (1996), if classroom instruction emphasizes only pen and paper algorithm to solve addition and subtraction problems, then, it is likely for students to execute given procedures without understanding their meaning. On the other hand, if instruction supports students' understanding, then understanding and skills may develop together. Therefore, it might be necessary to explore the associations between understanding and procedures in different instructional environments in order to get better sense of the interaction between understanding and skills.

One study conducted by Hiebert and Wearne (1996) focused on the interaction between conceptual understanding and procedural skills. They followed 72 first-grade students over three years of school. They provided an alternative instruction on place value and multidigit addition and subtraction and a conventional textbook instruction on the same topics to the experimental and control group, respectively. Based on their findings, the authors proposed that students who demonstrated conceptual understanding would be more likely than their peers to invent their own procedures and adjust old ones to solve new problems. Also, Heirdsfield and Cooper (2004b) carried out a study with seven third grade students. Based on their in-depth interviews with the seven students, the authors concluded that lack of procedural understanding contributed inaccuracies in mental computation and the inability to use alternative strategies. In addition, studies show that students who benefit from effective mental computation skills can use a variety of strategies in different situations and make flexible use of the structure of the number systems because they were disposed to making sense of mathematics (Cooper et al., 1996; Sowder, 1992, 1994; Heirdsfield & Cooper, 2004a, 2004b). In other words, flexible computers are capable of selecting the most efficient way to solve addition and subtraction problems since they have conceptual understanding of operations, numeration, number facts, and estimation. Regarding to the literature cited above, it can be concluded that students in primary grades should develop number sense in order to be successful in mental computation or vice versa. Thus, the next mystery point should be whether it is necessary to teach mental computation skills to children directly or encourage children to discover those skills by themselves.

TO TEACH OR NOT TO TEACH

As noted above, researchers have noticed the importance of mental computation, and have been exploring its effects on students' success in mathematics. Studies found that mental computation for addition and subtraction of multidigit numbers plays a significant role on teaching children how numbers work, how to make decisions about procedures, and how to create different strategies to solve math problems (Cobb & Merkel, 1989; Klein & Beishuizen, 1994; Maclellan, 2001; Reys, 1984, 1985; Sowder, 1992). Therefore, it is critical to find out how primary school children learn mental computation skills: Should teachers directly teach them those skills or should teachers encourage students to develop their own skills? The following section aimed to explore the literature in order to uncover the answer for these questions.

Directly Teaching Metal Computation Strategies

The National Assessment of Educational Progress (NAEP) is a national assessment that evaluates what students know and can do in various subject areas and that is one of the projects of Department of Government (http://nces.ed.gov/nationsreport card). NAEP mathematics scores are reported for grades 4 and 8 on a 0–500 scale. There are also three achievement levels: basic, proficiency, and advanced with the cut-off scores of 214, 249, and 282, respectively. Two decades ago, the results showed that elementary school children had very weak number skills (Carpenter, Coburn, Reys, & Wilson, 1976). Unfortunately, in 2005, the results of the NAEP reaffirmed this conclusion. According to the test scores, although fourth graders' national average mathematics scores increased by 25 points from 1990 to 2005 (NAEP, 2005), their average score was just above the basic level. The test scores may imply that children are not successful in building relationships between their conceptual knowledge and the procedures they are taught in school.

Also, recent studies show that although there are different types of strategies for mental addition and subtraction, the literature suggests that students often do not use their strategic knowledge to solve problems (Heirdsfield & Cooper, 2004a, 2004b; Siegler & Campbell, 1989). For example, many students keep counting on their fingers although they can retrieve the answer from memory or they simply use mental image of pen and paper algorithm. Hiebert and Lefevre's study (as cited in Sowder, 1992) explains the possible reasons for this deficit. According to their conclusion, the first reason is the deficit in knowledge base. In other words, children who do not understand, for instance, place value may not fully understand the decimal numbers. The second reason is the difficulty in encoding relationships that are obvious to the adults. The final one is the belief that information learned out of school should be kept separate from the information learned in school. Also, Blote et al. (2000) summarized the studies that focused on the factors that may influence children's strategy preferences: a) children may not know when and how a certain strategy can be implemented, b) they may not recognize the value of a strategy and the relative usefulness compared to other strategies, c) they may not be able to manage how much effort they have to spend on a strategy, d) the classroom context may be a very important factor in influencing students' strategic behavior. Considering also Hiebert and Wearne's emphasis on instruction (1996), mathematics instruction might be the critical point for children's selection of computation strategies.

Similar to Hiebert and Wearne (1996), Sowder concluded his previous research and noted that "there is evidence that instruction on mental computation can lead to both increased understanding of number and flexibility in working with numbers" (1992, p. 14). In their study, Blote and her colleagues (2000) investigated the relationship between the mental strategies the second graders preferred to use and the value those children gave to the strategies they chose. The participants received instruction that was intended to teach, first, the N10 strategy, and then the 1010 strategy. The authors reported that at the beginning of the study children used and valued the N10 strategy. However, after the introduction of the 1010 strategy, use and valuing of the 1010 strategy increased. This study may be considered as evidence that shows how direct instruction of computation strategies affects students' use of strategies.

In another study, Cooper et al. (1996) summarized the studies focusing on teaching implications for mental strategies. Based on their review of the literature, Cooper and his colleagues noted that it is necessary to reduce the amount of time that is spent on teaching pen-and-paper algorithm and to begin explicit exploration of mental algorithms. Also, they noted that low performing mental computation students should be identified and directly taught mental strategies. More specifically, Klein and Beishuizen (1994) summarized the studies investigated the mental computation strategies and whether those strategies should be taught. Based on those studies, Klein and Beishuizen suggested that "the didactic order should then be: first N10 to enhance mental (non-column) arithmetic and, much later, 1010 as a transition to written (column) arithmetic" (p. 127, 1994). In another study, Klein and his colleagues (1998) summarized the literature and noted that students who are less capable of solving addition and subtraction problems may need to receive structured instruction in which the teacher help them construct mental strategies to solve problems.

In their pilot study, Klein and Beishuizen (1994) observed classrooms that differed in didactic sequence and instructional design. In some classrooms, students were introduced the N10 strategy at the beginning of the study and flexibility was encouraged after students became more sensitive to the N10 strategy. Here, by flexibility, the authors mean to be able to use N10-like strategies (e.g., N10C and A10). In the other classrooms, both the N10 strategy and flexibility were emphasized from the beginning by focusing on children's informal strategies. Their analyses showed that out of 176 students, only 16 students preferred to use the 1010 strategy in order to solve subtraction problems with multidigit numbers. On the other hand, 104 students used the N10 strategy for the same type of questions. The number of the students who used N10 and 1010 strategies for addition problems were 87 and 49, respectively. This study may suggest that elementary school students may not attempt to invent different mental computation strategies if they are taught a specific mental computation method to solve addition and subtraction problems with multidigit numbers.

Moreover, Reys clearly states that "mental computation should be a visible part of an elementary mathematics program" (1984, p. 550). According to him, there are five widely accepted reasons for teaching mental computation. The reasons he addressed are:

> (1) it is a prerequisite for successful development of all written arithmetic algorithms; (2) it promotes greater understanding of the structure of numbers and their properties; (3) it promotes creative and independent thinking and encouraging students to create ingenious ways of handling numbers; (4) it contributes to the development of better problem-solving skills; and (5) it is a basis for developing computational estimation skills. (p. 549)

Encouraging Students to Develop Individual Mental Computation Strategies

On the other hand, there is a growing body of research that shows that children can develop their own efficient and skilled strategies spontaneously without instruction (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998; Heirdsfield, 2000; Kamii, Lewis, & Jones, 1991; Kamii, Lewis, & Livingston, 1993). Kamii et al. (1993) stressed that children can invent their own strategies if they are not taught a specific way to solve addition and subtraction problems with multidigit numbers. They posited that teachers should create a classroom environment where students feel comfortable enough to share their ideas/solutions with their peers; and then, by writing a few questions on the board, they should ask students to find a quick and easy way of solving those problems. If no one attempts to find a solution, that means, according to Kamii, Lewis, and Livingston, the questions on the board are too hard for the class; therefore, classroom teachers should give easier questions to students to solve. In addition, Heirdsfield (2000) conducted in-depth interviews with 13 third grade students in order to explore the reasons that lead some children to be better at addition and subtraction mental computation than others. She found that children who invented their own strategies were more accurate and showed more number sense than the teacher taught strategies; therefore, Heirdsfield inferred that students should be encouraged to formulate their own strategies as solving addition and subtraction problems with multidigit numbers.

Carpenter et al. (1998) conducted a longitudinal study in order to investigate the development of children's understanding of multidigit number concept. There are three main characteristics of their study that need to be considered. The first characteristic is the discourse which allowed students and teachers to discuss students' solutions in the classroom. The second one is the materials that were used in classroom. According to the authors, children were allowed to use tens-blocks and other base-ten materials to solve the problems as long as they wanted. The last one is the problems children were given to solve. The participant students were given word problems involving regrouping of multidigit numbers. One of the advantages of word problems is that children tend to use more mental strategies if they are provided word problems rather than number exercises (Heirdsfield & Cooper, 2004a). Those characteristics might be the most important factors that influence children's invention of mental computation strategies. Their results showed that children could invent strategies for adding and subtracting without instruction. In addition, Carpenter and his colleagues noted that if children were taught the computation strategies directly, "there would be a danger that children would learn them as rote procedures in much the way that they learn standard algorithms" (1998, p. 14).

CONCLUSION

Today researchers explore the importance of mental computation and its effects on students' success in mathematics. According to the literature, mental computation helps children understand how numbers work, how to make decisions about procedures, and how to create different strategies to solve math problems. Also, the association between conceptual understanding and mental computation skills has been addressed. Specifically, studies concluded that students who demonstrated conceptual understanding are more likely to develop deeper understanding of mental computation skills. Although researchers agree on the importance of mental computation skills and stressed the relationship between conceptual understanding and procedural skills, they debate how to help students develop these skills. The limited empirical study investigated how different instructional settings affect students' accuracy and flexibility in employing a variety of strategies to a problem. However, despite the existing studies, there still exist three mysterious points that are related to students' use of different mental calculation strategies in a variety of settings and their conceptual understanding of those strategies. The first point is that whether children should be given a chance to discover those strategies based on their own knowledge and natural skills. The second point is that whether children need to be taught the strategies and to be then allowed using the strategies that they feel more comfortable to solve addition and subtraction problems. If children are taught directly or allowed to invent their own strategies, then the final point is to clarify how either circumstance affects children's flexibility in and conceptual understanding of solving addition and subtraction problems with multidigit numbers. Therefore, further studies should focus on addressing these questions in order to improve the quality of mathematics education.

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