Laboratory 4
Logistic Equation Exploration

Introduction

This laboratory exercise focuses on the logistic growth model. Through this exercise, you will learn to encode and solve the continuous time version of the model using both the exact analytical solution as well as two numerical approximation techniques: the Euler method and *lsoda*. You will also practice using a generalized form of the logistic control function and encounter Shaffer's extension of the logistic model to consider maximum sustainable yield. In addition, you will further develop your scientific programming skills in R .

Please report your findings in the form of a short report in which you present evidence that you accomplished each of the modeling tasks listed. Please make sure to address any questions asked and include any figures or tables necessary. Your short report will be due at our next lab meeting.

Tasks

Task 4.1: Diagram the model

Create a diagram of the logistic growth model using the Forrester symbols that we discussed in class. Make sure to clearly label each part.

Task 4.2: Estimate Population Projections

Modify the R scripts you used for the last laboratory to project the future population size of a population growing according to the logistic growth model. For your nominal runs, assume that $r = 0.1$, $N_0 = 5$, $K = 100$, and time from 1 to 100 days. Please compare the solution of this model using the analytically exact solution (see lecture notes), the Euler solution with $dt = \{1, 0.1, 0.01\}$, and the *lsdoa* algorithm in R . Plot all 5 solutions on one graph and calculate RMSEP for each numerical approximation. Finally, describe any differences you observed in how the numerical error accumulates in this model when compared to the exponential growth model.

Task 4.3: Modifying the Logistic Control Function

Next we will consider how the generalized logistic control function can be used. Recall that this function is

$$f(N) = \left(1 - \frac{N}{K}\right)^b. \tag{1}$$

Where N is the population density, K is the population carrying capacity, and b is a parameter that changes the shape of the density dependent response.

Task 4.3.1: Control Function Plot

Create a plot that shows how the control function changes with population density N when b, the shape parameter, has values of $\{0.5, 0.75, 1, 1.25, 1.5\}$. Assume N changes from 0 to 200, that the carrying capacity of the system is 100, and the initial population size is 5.

Please plot all five on the same graph to facilitate comparison. Explain what the b parameter does and consider an ecological rational for why you might want it to be different from 1?

Task 4.3.2: Population Projections

Use lsoda to estimate the population sizes from $t = 1$ to $t = 100$ in the model using the generalized logistic when $b = \{0.5, 1, 1.5\}$. Again, please plot these on the same graph for comparison. How does $b \neq 1$ change the population dynamics?

Task 4.4: Schaffer Equation

M.B. Schaefer extended the logistic model to further consider the effect of fishing on fish stocks. He assumed that fish harvest was proportional to abundance and modified the logistic equation as

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - FN, \tag{2}$$

where F is the mortality from fishing pressure. Fishing mortality is often described as a function of the fishing effort E and the effectiveness q of catching and removing the fish. Thus, $F = qE$.

We know from our initial analysis of the logistic equation that in the absence of fishing, the maximum growth rate is achieved when $N = K/2$ and that the growth rate at this maximum is $rK/4$. In the fisheries context, the maximum growth rate is the maximum sustainable yield (MSY) because if we could maintain the stock at this level and harvest the production, it generates the greatest yield. The fish population size that generates the MSY is termed the maximum net productivity (MNP).

What happens when we add fishing? As long as the harvest equals the biological production, the stock will remain the same size. We can use the model (equation 2) to find this equilibrium population size as follows.

To find the population size at equilibrium, we first set the differential equation to zero and solve for N^* as follows

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - qEN \tag{3}$$

$$0 = rN\left(1 - \frac{N}{K}\right) - qEN \tag{4}$$

$$qEN = rN\left(1 - \frac{N}{K}\right) \tag{5}$$

$$qE = r\left(1 - \frac{N}{K}\right) \tag{6}$$

$$\frac{qE}{r} = 1 - \frac{N}{K} \tag{7}$$

$$\frac{N}{K} = 1 - \frac{qE}{r} \tag{8}$$

$$N^* = K\left(1 - \frac{qE}{r}\right) \tag{9}$$

We can now substitute this equilibrium population size into the fishing function to determine the yield (Y^*).

$$= qEN^* \tag{10}$$

$$Y^* = qEK\left(1 - \frac{qE}{r}\right). \tag{11}$$

This is a parabola with a maximum occurring at $E^* = r/2q$.

Task 4.4.1

As biologists and resource managers, we might want to describe how the yield changes with fishing effort. Create a plot of selected values of E versus the steady state yield Y^* (equation 11). Assume $q = 1$, $K = 100$, and $r = 0.1$.

Task 4.4.2

To better understand the relationship between the biological growth rate, harvest rate, and population density, please create a figure with two plots. In each, plot the growth rate versus population density for the logistic growth function. To each plot, add a line showing how the harvest rate (qEN) changes with population density. Recall that where these two curves intersect indicates the intersection of the equilibrium population density and its equilibrium yield. In the first plot, show a case where the population is under fished. In the second plot, show a case where the population is over fished.

References

Mangel, M. 2006. The theoretical biologist's toolbox: quantitative methods for ecology and evolutionary biology. Cambridge.