Math Warm Up
Solutions

BIO534

Exercise 1.1

Objective Show that the rate of flow of energy when using a generalized feeding strategy is

$$R_g = \frac{E_1\lambda_1 + E_2\lambda_2}{1 + h_1\lambda_1 + h_2\lambda_2} \tag{1}$$

Given the following:

Symbol	Definition	Unit
S	time spent searching	[T]ime
H	time spent handling	T
$T = S + H$	total time	T
λ_i	predator encounter rate of prey type i	T^{-1}
E_i	energy gained by food type i	[C]al
R_g	rate of energy gain by generalizing	$\mathrm{C\ T}^{-1}$

Intermediate Calculations with biological relevance to help us solve our problem.

Equation	Definition	Unit
$\lambda_i S$	number of items encountered while searching	prey number
$E_i\lambda_i S$	energy gain from food type i	C
$H_i = h_i\lambda_i S$	handling time	T
$E_1\lambda_1 S + E_2\lambda_2 S$	total energy gain	C

Solution

$$\begin{aligned}
R_g &= \frac{\text{total energy gain}}{\text{time}} && (2)\\
&= \frac{E_1\lambda_1 S + E_2\lambda_2 S}{T} && (3)\\
&= \frac{E_1\lambda_1 S + E_2\lambda_2 S}{S + h_1\lambda_1 S + h_2\lambda_2 S} && (4)\\
&= \frac{S(E_1\lambda_1 + E_2\lambda_2)}{S(1 + h_1\lambda_1 + h_2\lambda_2)} && (5)\\
&= \frac{E_1\lambda_1 + E_2\lambda_2}{1 + h_1\lambda_1 + h_2\lambda_2} && (6)
\end{aligned}$$

Exercise 1.2

Objective show that

$$R_s > R_g \implies \lambda_1 = \frac{E_2}{E_1 h_2 - E_2 h_1} \tag{7}$$

Solution "There are only two cases where inequalities differ from equations: when you multiply or divide both sides by a negative number (or variable with a negative coefficient). In these two cases, you need to flip the inequality sign to maintain the proper relationship." (from http://wps.prenhall.com, accessed Aug. 2010).

Given our problem definition, all of our equation elements must be positive because of their physical interpretation. Thus, we can solve this equation as if the inequality was an equality.

$$\begin{aligned}
R_s &> R_g && (8)\\
\frac{E_1\lambda_1}{1+h_1\lambda_1} &> \frac{E_1\lambda_1 + E_2\lambda_2}{1+h_1\lambda_1+h_2\lambda_2} && (9)\\
E_1\lambda_1(1+h_1\lambda_1+h_2\lambda_2) &> (E_1\lambda_1+E_2\lambda_2)(1+h_1\lambda_1) && (10)\\
E_1\lambda_1 + E_1h_1\lambda_1^2 + E_1\lambda_1\lambda_2h_2 &> E_1\lambda_1 + E_2\lambda_2 + E_1h_1\lambda_1^2 + E_2h_1\lambda_1\lambda_2 && (11)\\
E_1\lambda_1\lambda_2h_2 &> E_2\lambda_2 + E_2h_1\lambda_1\lambda_2 && (12)\\
E_1\lambda_1\lambda_2h_2 &> \lambda_2(E_2 + E_2h_1\lambda_1) && (13)\\
E_1\lambda_1h_2 &> E_2 + E_2h_1\lambda_1 && (14)\\
E_1\lambda_1h_2 - E_2h_1\lambda_1 &> E_2 && (15)\\
\lambda_1(E_1h_2 - E_2h_1) &> E_2 && (16)\\
\lambda_1 &> \frac{E_2}{E_1h_2 - E_2h_1} && (17)
\end{aligned}$$

Exercise 1.3

This problem was to investigate the effect of a slight change in functional form of an equation. This is something that could be solved by hand or using excel, but here I will present my solution using R. This is a very generic problem that we will encounter many times in class.

My R script to solve this problem follows. It generates the plot shown in Figure 2.

```
# Exercise 1.3
# Math Warm up. Mangel 2006
# Borrett | Aug 2010
# ------------------

# ----- INPUT ----------
# define paramters
t <- 0:100   # time
a <- 1       # shape parameter
b <- 100       # shape parameter

# ----- ACTION ----------
# solve equations
g1 <- a * t / (b + t)
g2 <- a * t^2 / (b + t^2)

# ----- OUTPUT ----------
# PLOT

par(las = 1)  # change plotting paramters

# create plot
plot(t, g1,
     pch = 20,
     col = "blue",
     ylim = c(0,a),
     type = "b",
     cex = 1.5,
     ylab = "functional response",
     xlab = "time",
     main = "Exercise 1.3")

# add second line to the existing plot
points(t, g2,
       pch = 19,
       col = "green",
       type = "b",
       cex = 1.05)

# create the figure legend
legend(60, 0.8,
       legend = c("g1","g2"),
       col = c("blue","green"),
       pch = c(20,19),
       bty = "n",
       cex = 2)
```

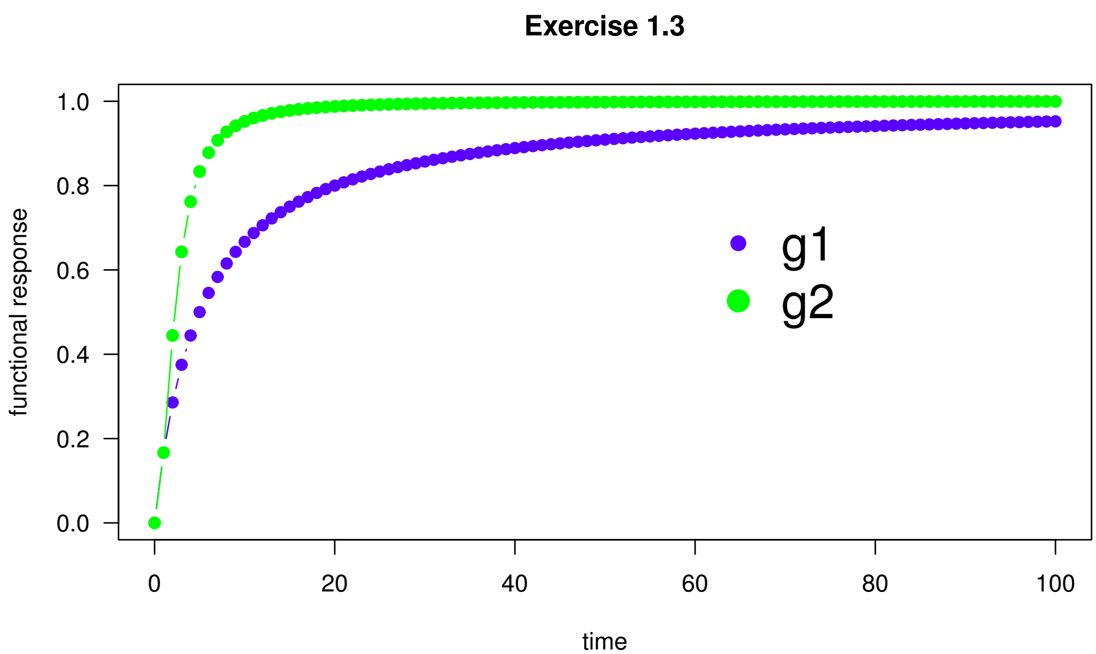

Figure 1: Plot of two functions for exercise 1.3

Exercise 1.4

Objective Show that the optimal egg size is $x_{opt} = (c(a+1))^{1/a}$ and that for the values from Einum and Feleming $x_{opt} = 0.1244g$.

Solution We can find the optimal egg size by setting the derivitive of $R(g, x)$ with respect to x equal to zero and solve for x.

Given equation 18

$$\frac{dR(g,x)}{dx} = g/x(1 - cx^{-a}) \tag{18}$$

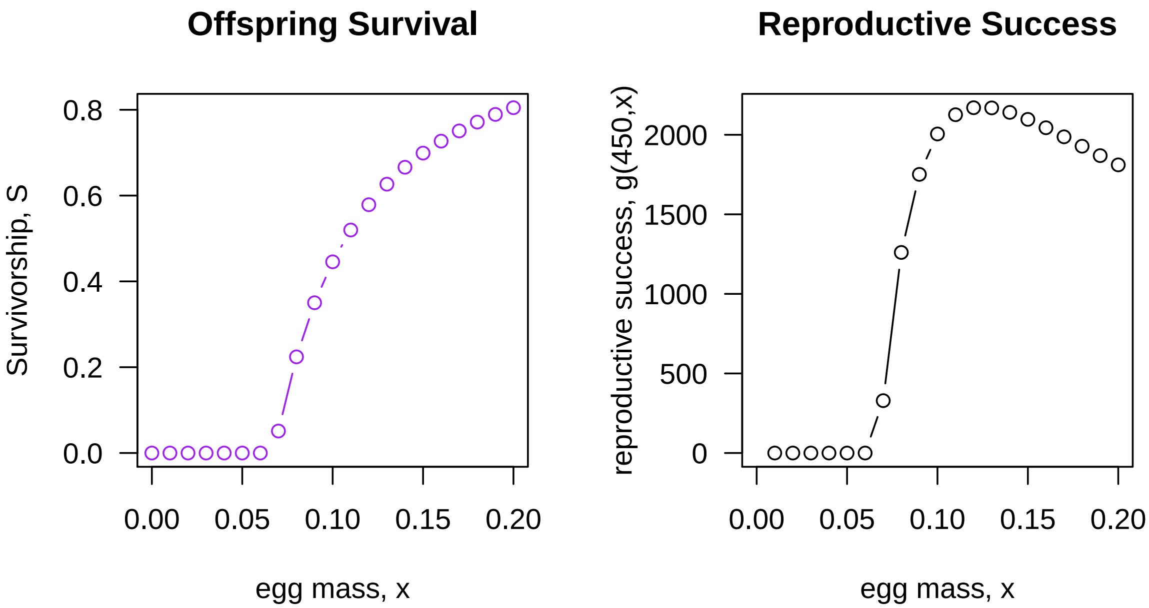

Figure 2: Plot of two functions for exercise 1.4

Solve

$$\frac{dR(g,x)}{dx} = g/x(1 - cx^{-a}) \tag{19}$$

$$= gx^{-1}(1 - cx^{-a}) \tag{20}$$

$$= gx^{-1} - cgx^{-a-1} \tag{21}$$

$$= -gx^{-2} - (-a-1)cgx^{-a}x^{-2} \text{ [differentiate with power rule]} \tag{22}$$

$$0 = -gx^{-2} - (-a-1)cgx^{-a}x^{-2} \tag{23}$$

$$gx^{-2} = (a+1)cgx^{-2}x^{-a} \tag{24}$$

$$1 = c(a+1)x^{-a} \tag{25}$$

$$(c(a+1))^{-1} = x^{-a} \tag{26}$$

$$(c(a+1))^{1/a} = x \text{ [take a-th root of both] sides} \tag{27}$$

For Einum and Fleming, we are given that $x_{min} = 0.0676g$, and $a = 1.5066$. Then, $c = (x_{min})^a = (0.0676)^{1.5066} = 0.01726624$ and

$$x_{opt} = (c(a+1))^{1/a} \tag{28}$$

$$= 01726624(1.5066+1)^{1/1.5066} \tag{29}$$

$$= 0.124405 \tag{30}$$

To create a visualization for this problem as shown in Mangel, I created the following R script.

```
# Exercise 1.4
# Math Warm up.  Mangel 2006
# Borrett | Aug. 2010
# ----------------------

# ------- INPUT ------------------
# independent variable
x <- seq(0, 0.2, by = 0.01)  # singel egg mass

# paramters
xmin <- 0.0676  # minimum viable egg mass
a <- 1.5066     # shape parameter fit from data
g <- 450        # mother's gonadal mass


# ------- ACTION -----------------
# intermediate calculation to simplify
c <- xmin^a

# Calculate egg survivorship
S <- 1 - c * x^(-a)  # survivorship

S[S<0] <- 0  # survivial cannot be less than zero; find and replace
            # values less than zero with zero

R <- g/x * S  #


# ----------- OUTPUT ------------
pdf("mangel_1-4.pdf", height = 4, width = 7) # open PDF file in which to
                                             # write the figure

opar = par(las = 1, mfrow = c(1,2))  # plotting parameters

# create the plot

plot(x, S, type = "b",
     col = "purple",
     xlab = "egg mass, x",
     ylab = "Survivorship, S",
     main = "Offspring Survival")

plot(x, R, type = "b",
     main = "Reproductive Success",
     xlab = "egg mass, x",
     ylab = "reproductive success, g(450,x)")

dev.off() # close the PDF file

rm(opar) # rm changes to the plotting paramters

system("open mangel_1-4.pdf")  # an MAC or LINUX OS, this opens the PDF file
```