Homework 4: Solution

S.R. Borrett

BIO534

Chapter 1: Exponential Growth

Problem 1.2

Given $N(t) = 3000$, births = 400, and deaths = 150:

$$N(t+1) = N(t) + births - deaths \tag{1}$$
$$N(t+1) = 3250 \tag{2}$$

Then,

$$\lambda = N(t+1)/N(t) = 3250/3000 = 1.08433$$

and

$$r = e^{\lambda} = 0.08$$

We then project the poulation out 6 months as follows:

$$N(6) = 3000 * e^{0.08*6} \tag{3}$$
$$N(6) = 4848 \tag{4}$$

Problem 1.4

In this problem we are considering a population that is increasing my 12% per year. The trick is to realize that this 12% increase means a discrete time growth rate of $\lambda = 1.12$.

Given this, we can find $r = ln(\lambda) = 0.1133$ and then solve the doubling time equation (1.3 in Gotelli) to determine that the doubling time is 6.116 years.

Chapter 2: Logistic Growth

Problem 2.1

Given a population growing logistically with a growth rate of $r = 0.1$ and a carrying capacity of $K = 500$, what is the maximum growth rate?

Recall from figure 2.3 that the maximum growth rate is achieved when $N = K/2 = 250$. We substitute this into the logistic equation and solve for the growth rate.

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \tag{5}$$
$$= 0.1 * 250 * (1 - 0.5) \tag{6}$$
$$= 25 * 0.5 \tag{7}$$
$$= 12.5 \tag{8}$$

Problem 2.2

Given that the manager is keeping the fish population at 500 individuals to maximize yield, then $K/2 = 500$ and $K = 1000$.

We can then find the instantaneous growth rate when $r = 0.005$ as $\frac{dN}{dt} = 0.005 * 500(1 - 500/1000) = 1.25$. If we add 600 fish, this alters the instantaneous growth rate to $\frac{dN}{dt} = 0.005 * 1100(1 - 1100/1000) = \mathbf{-0.55\ fish/day}$. One quick sanity check is that this new growth rate is negative, which is what we would expect since the population exceeded its carrying capacity.

Problem 2.3

Plot the different density dependent birth and growth rates on the same graph (Fig. 1).

```
> N = 1:50
> b = 0.10 + 0.03 * N - 0.0005 * N^2
> d = 0.20 + 0.01* N

> # PLOT SOLUTIONS
> opar <- par(las=1,mar=c(4,4,1,1),oma=c(0,0,0,0))
> plot(N,b,ylim=c(0,1),type="l",col="blue",lwd=3,cex=0.5,
+      ylab="rates")
> points(N,d,col="red",type="l",lwd=3,lty=3)
```

This example is different from the logistic growth in that the birth rate is not just a linear dependence on population density; instead it has a parabolic density dependency. This is an example of how to mathematically model an **Allee effect**.

Note the two intersections of the death rate line and the birth rate parabola. These are mathematical equilibrium. The one on the left (near 6) is unstable and the one on the right (near 36) is stable. The left equilibrium builds into the model the failure of the population if its below the minimum population size.

Chapter 3: Structured Population Growth

Problem 3a

```
> # Enter Data and Build Initial Table
> age=0:3
> sx = c(500,400,40,0)
> bx = c(0.0,2.5,3.0,0.0)
> d <- data.frame(age,sx,bx)
> # new life table claculations
```

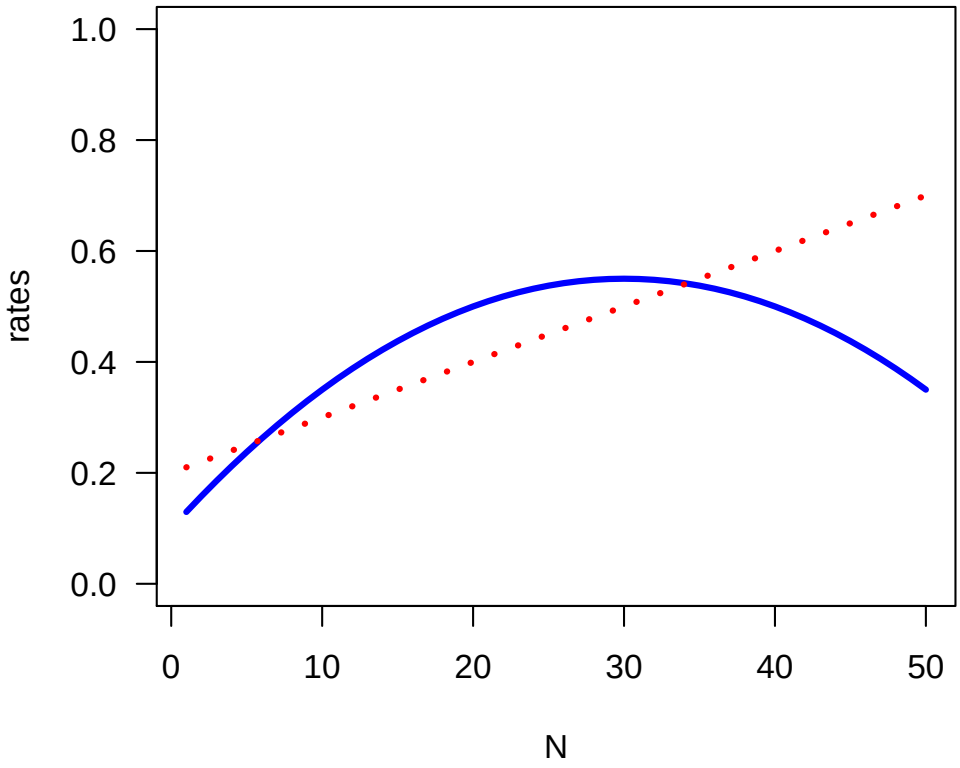

Figure 1: Intersection of density dependent birth and death rates as defined in problem 2.3. The birth rate is a parabola in blue and the death rate is a linear function of density shown in red. The intersections indicate equilibrium points.

```
> lx = sx/sx[1]
> gx = c(lx[2:4] / lx[1:3] , NA)
> lxbx = lx*bx
> Ro = sum(lxbx)
> G = sum(lx*bx*age)/sum(lx*bx)
> r.est = log(Ro)/G
> ## trial and error to find growth rate correction factor
>
> # build function based on Euler equation
> r.corr = function(r){
+   r.corrected = sum( exp(-r * age) * lxbx)
+   return(r.corrected)
+ }
> r.corr(r.est)

[1] 1.021245

> r.corr(r.est+0.01)

[1] 1.010532

> r.corr(r.est+0.02)
```

```
[1] 0.9999374

> r.corr(r.est+0.019)

[1] 1.000992

> r.corr(r.est+0.018)

[1] 1.002047

> r.corr(r.est+0.0195)

[1] 1.000464

> r.corr(r.est+0.01994)

[1] 1.000001

> r.corr = r.est + 0.01994  # final value
> show(r.corr)

[1] 0.7483698

> # Results Summary
> life.table <- data.frame(age,sx,bx,lx,gx,lxbx,exp(-r.est*age)*lxbx,exp(-r.corr*age)*lxbx)
> show(life.table)

  age  sx  bx   lx  gx lxbx exp..r.est...age....lxbx exp..r.corr...age....lxbx
1   0 500 0.0 1.00 0.8 0.00               0.00000000                0.00000000
2   1 400 2.5 0.80 0.1 2.00               0.96533254                0.94627445
3   2  40 3.0 0.08 0.0 0.24               0.05591202                0.05372612
4   3   0 0.0 0.00  NA 0.00               0.00000000                0.00000000

> rates = c("Ro" = Ro,
+     "G" = G,
+     "r_est" = r.est,
+     "r_corrected" = r.corr,
+     "correction factor" = 0.01994)
> show(rates)

              Ro                G            r_est       r_corrected
       2.2400000        1.1071429        0.7284298        0.7483698
correction factor
       0.0199400

>
>
```

Problem 3b

The **stable-stage distribution** is calculated by multiplying the survivorship-schedule lx by the exponential growth solution as

$$lx * e^{-r*age}$$

and then dividing the vector by its summation.
Using $r = 0.749$ this generates the following

```
> cx = lx * exp(-0.749 * age)
> cx <- cx/sum(cx)
> show(cx)

[1] 0.71625158 0.27093743 0.01281098 0.00000000
```

Gotelli says that the **reproductive value distribution** can be calculated by equation 3.16. However, I could not figure out how to implement this. Instead, I created the Leslie matrix model and found the right hand eivenvector.

```
> P=0; for(i in 2:length(lx)){ P[i] = lx[i]/lx[i - 1]} # survivial probabilities
> F=P *bx  # Fecundities
> P <- P[2:4]
> F <- F[2:4]
> #
> # build Leslie Matrix
> A <- matrix(0,nrow=length(age)-1,ncol=length(age)-1) # initialize matrix
> A[1,] <- F
> A[2,1] <- P[1]
> A[3,2] <- P[2]
> #
> # --- Analysis ---
> # get right hand eigenvector (stable stage dist)
> e <- eigen(A)
> rev1 <- e$vectors[,1]
> rev1 <- rev1/sum(rev1)
> show(rev1) # stable stage distribution

[1] 0.71611782 0.27105745 0.01282473

> # get left hand eivenvectors (reproductive value)
> e <- eigen(t(A)) # transpose A
> lev1 <- e$vectors[,1]
> lev1 <- lev1/lev1[1]
> show(lev1) # stable stage distribution

[1] 1.0000000 0.1419411 0.0000000
```