Splitting Whist Tournament Designs

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Abstract

In this paper a new specialization of whist tournament designs is introduced. A splittable whist tournament design on $v$ players, $\text{SWh}(v)$, is a whist tournament design that has the property that the games can be partitioned into two sets $A, B$ in such a way that for every pair of players, say $\{x, y\}$, $x$ opposes $y$ once in $A$ and once in $B$. Let $p$ denote a prime of the form $p = 2^k t + 1$, $k \geq 3$, $t$ odd. For $k = 3$ we show that $\text{SWh}(p)$ exist for all $p$ except, possibly, $p = 41$. For $k \geq 4$, $\text{SWh}(p)$ exist for all $p < 100000$ except, possibly, $p = 17, 257, 65537$. 
1 Introduction

A whist tournament on \( v \) players, denoted \( \text{Wh}(v) \), is a \((v, 4, 3)\) (near) resolvable BIBD that possesses the following properties. Each block, \((a, b, c, d)\), of the BIBD is called a whist game. For such a game, the partnership \( \{a, c\} \) opposes the partnership \( \{b, d\} \). The \textbf{whist conditions} require that every player partners every other player exactly once and every player opposes every other player exactly twice. The (near) resolution classes of the design are called rounds of the \( \text{Wh}(v) \). It has been known since the 1970s that \( \text{Wh}(v) \) exist for all \( v \equiv 0, 1 \pmod{4} \) [1].

A \( Z \)-cyclic whist design is such that the players are elements in \( Z_m \cup C \) where \( m = v \), \( C = \emptyset \) when \( v \equiv 1 \pmod{4} \) and \( m = v - 1, C = \{\infty\} \) when \( v \equiv 0 \pmod{4} \). It is further required that the rounds be cyclic in the sense that the rounds can be labeled, say, \( R_1, R_2, \ldots \) in such a way that \( R_{j+1} \) is obtained by adding \( +1 \pmod{m} \) to every element in \( R_j \). \( R_1 \) is typically called the initial round. We shall denote the initial round as \( IR \).

\textbf{Definition 1.1} A \( \text{Wh}(v) \) is said to be \textbf{splittable} if and only if the games can be partitioned into two sets \( A \) and \( B \), called the partition sets, such that in each round half the games are in \( A \) and the other half are in \( B \). In each partition set every player opposes every other player exactly once.

\textbf{Lemma 1.2} If a \( \text{Wh}(v) \) is splittable then \( v \equiv 1 \pmod{8} \).

\textit{Proof:} Since the \( \text{Wh}(v) \) is splittable, the total number of games, \( \frac{v(v - 1)}{4} \), must be even. Thus \( v(v - 1) \) is a multiple of 8. If \( v \equiv 0 \pmod{8} \) then every player \( x \) plays an odd number of games, contradicting the assumption that \( x \) opposes every other player exactly once in each partition set. Therefore, \( v \equiv 1 \pmod{8} \).

Splittable whist tournament designs on \( v \) players will be denoted by \( S\text{Wh}(v) \).

\textbf{Remark 1.3} There is no \( Z \)-cyclic \( S\text{Wh}(17) \). This result is established by exhaustion via examination of partner and opponent differences. The existence of a non \( Z \)-cyclic \( S\text{Wh}(17) \) is open.
Lemma 1.4 Let \( k \) be an integer greater than or equal to 3. Let \((a_1, a_2, a_3, a_4)\) be a quadruple of positive integers and set \( b_i = a_i \pmod{4} \). If \((b_1, b_2, b_3, b_4)\) is a complete set of residues modulo 4, then \( S_1 = \{b_i + 4s : 1 \leq i \leq 4, 0 \leq s \leq 2^{k-2} - 1\} \) is a complete set of residues modulo \( 2^{k-1} \). Similarly the set \( S_2 = \{b_i + 4s + 2 : 1 \leq i \leq 4, 0 \leq s \leq 2^{k-3} - 1\} \) is a complete set of residues modulo \( 2^{k-1} \).

Proof: If one assumes the contrary for either set then there exist two elements, \( b_i, b_j \) such that \( b_i = b_j \), a contradiction. \( \blacksquare \)

Let \( p \) denote a prime of the form \( p = 2^k t + 1 \) where \( k \geq 3 \) and \( t \) is odd. Let \( r \) be a primitive root of \( p \). Set \( d = 2^k, m = 2^{k-1}, n = 2^{k-2} \). Throughout the remainder of this study \( d, m \) and \( n \) are understood to be given by these formulæ. If \( y \in \mathbb{Z}_p \setminus \{0\} \), then there exists a unique \( i \) such that \( y = r^i \). Set \( \text{ind}(y) = i \).

Definition 1.5 Let \((a, b, c, d)\) be a game in a \( \mathbb{Z} \)-cyclic whist tournament on \( p \) players. The differences \( a - b, c - d, a - d, b - c \) are called the split opponent differences.

We will be interested in the cyclotomic classes, of order \( m \), associated with the split opponent differences. Now \(-1 = r^{(p-1)/2} = r^m \) hence \( a - b \) and \( b - a \) are in the same cyclotomic class and our convention will be to choose the positive difference when referring to the split opponent differences.

Theorem 1.6 Let \( p \) denote a prime of the form \( p = 2^k t + 1 \) where \( k \geq 3 \) and \( t \) is odd. Let \( r \) be a primitive root of \( p \). The set of games \((1, x, x^m, -x) \otimes r^{d+2j}, 0 \leq i \leq t-1, 0 \leq j \leq n-1\) forms the initial round of a \( \mathbb{Z} \)-cyclic SWh(p) if \( (1) \) \( x \) is a non-square in \( \mathbb{Z}_p \), \( (2) \) \( x^m - 1 \) is a square in \( \mathbb{Z}_p \) and \( (3) \) the quadruple of integers \((a_1 = \text{ind}(x-1), a_2 = \text{ind}(x^m + x), a_3 = \text{ind}(x+1), a_4 = \text{ind}(x^m - x))\) satisfies the hypotheses of Lemma 1.4.

Proof: Since \(-1 = r^{(p-1)/2}\), these games will satisfy the partner whist condition if the product \( 2x(x^m - 1) \) is a non-square in \( \mathbb{Z}_p \). Given that 2 is a square in \( \mathbb{Z}_p \), hypotheses (1) and (2) guarantee that the partner condition is satisfied. Let us call the set of games \((1, x, x^m, -x) \otimes r^{2j}, 0 \leq j \leq n-1\) the base games for the initial round. These base games can
be split into two sets $B_1 = (1, x, x^m, -x) \otimes r^{4j}, 0 \leq j \leq 2^{k-3} - 1$ and $B_2 = (1, x, x^m, -x) \otimes r^{4j+2}, 0 \leq j \leq 2^{k-3} - 1$. Hypothesis (3) guarantees that, for each of $B_1, B_2$ the split opponent differences form a SDR for the cyclotomic classes of order $m$. Now, the set of initial round games can be split into two sets $IR_1 = (1, x, x^m, -x) \otimes r^{di+4j}, 0 \leq i \leq t-1, 0 \leq j \leq 2^{k-3} - 1$ and $IR_2 = (1, x, x^m, -x) \otimes r^{di+4j+2}, 0 \leq i \leq t-1, 0 \leq j \leq 2^{k-3} - 1$. Let $A$ denote the development modulo $p$ of $IR_1$ and let $B$ denote the development modulo $p$ of $IR_2$. Since the split opponent differences form a SDR for the cyclotomic classes of order $m$ for each of $B_1, B_2$ it easily follows that $A, B$ constitute partition sets for a SWh($p$).

**Example 1.7** A SWh(137). Here $k = 3, t = 17, d = 8, m = 4$ and $n = 2$. Thus each of the base sets will consist of one game and in each round the partition sets each contain 17 games. For the choice $r = 3, x = 24$ one obtains $B_1 = (1, 24, 99, 113)$ and $B_2 = (9, 79, 69, 58)$. Consequently $IR_1$ consists of the following 17 games

$$(1, 24, 99, 113), \quad (122, 51, 22, 86), \quad (88, 57, 81, 80),$$

$$(50, 104, 18, 33), \quad (72, 84, 4, 53), \quad (16, 110, 77, 27),$$

$$(34, 131, 78, 6), \quad (38, 90, 63, 47), \quad (115, 20, 14, 117),$$

$$(56, 111, 64, 26), \quad (119, 116, 136, 21), \quad (133, 41, 15, 96),$$

$$(60, 70, 49, 67), \quad (59, 46, 87, 91), \quad (74, 132, 65, 5),$$

$$(123, 75, 121, 62), \quad (73, 108, 103, 29),$$

and $IR_2$ is the following set of 17 games.

$$(9, 79, 69, 58), \quad (2, 48, 61, 89), \quad (107, 102, 44, 35),$$

$$(39, 114, 25, 23), \quad (100, 71, 36, 66), \quad (7, 31, 8, 106),$$

$$(32, 83, 17, 54), \quad (68, 125, 19, 12), \quad (76, 43, 126, 94),$$

$$(93, 40, 28, 97), \quad (112, 85, 128, 52), \quad (101, 95, 135, 42),$$

$$(129, 82, 30, 55), \quad (120, 3, 98, 134), \quad (118, 92, 37, 45),$$

$$(11, 127, 130, 10), \quad (109, 13, 105, 124).$$

Using symmetric differences [1] one can verify that the above 34 games, partitioned as indicated, constitute the initial round of a $Z$-cyclic SWh(137). Additional information pertaining to this design is presented in Section 2 below.

**Theorem 1.8** Let $p = 2^kt + 1, k \geq 3, t$ odd, be a prime. Splittable whist designs on $p$ players exist for $p < 100000$ except, possibly, for the following
cases:

- $k = 3$, $p = 41, 73, 89, 233, 281, 313$;
- $k = 4$, $p = 17, 113, 241, 337$;
- $k = 5$, $p = 97$;
- $k = 7$, $p = 641$;
- $k = 8$, $p = 257$;
- $k = 16$, $p = 65537$.

**Proof:** All of the claimed solutions satisfy the conditions of Theorem 1.6. These solutions were computer generated. A sampling of the solution data is given in Appendix I. The complete data set is available from any of the authors.

**Corollary 1.9** Let $p = 2^k t + 1$, $k \geq 3$, $t$ odd, be a prime. If $r$ is a primitive root of $p$ and $x$ is a non-square in $\mathbb{Z}_p$ such that there exists a SWh($p$) via Theorem 1.6 then $x$ together with the primitive root $r^{-1}$ also yields a SWh($p$) via Theorem 1.6.

**Proof:** Apply Lemma 2.4 in [2].

### 2 Some Examples

Throughout these examples Game 1 represents the game $(1, x, x^m, -x)$. Also opp$_1$ is the positive choice among $a - b$ and $b - a$. Similarly, opp$_2 = c - d$, opp$_3 = a - d$ and opp$_4 = b - c$.

$p = 137$

{r, x} = {3, 24}

Game 1: $(1, 24, 99, 113)$

$x^m - 1 = 98 = r^{94}$
opp\(_1\) = 23 = r\(^{125}\) and 125 \equiv 1 \pmod{4}
opp\(_2\) = 123 = r\(^{120}\) and 120 \equiv 0 \pmod{4}
opp\(_3\) = 25 = r\(^{14}\) and 14 \equiv 2 \pmod{4}
opp\(_4\) = 75 = r\(^{15}\) and 15 \equiv 3 \pmod{4}

\(p = 409\)
\(\{r, x\} = \{21, 21\}\)
Game 1: \((1, 21, 206, 388)\)
\(x^m - 1 = 205 = r^{326}\)
opp\(_1\) = 20 = r\(^{188}\) and 188 \equiv 0 \pmod{4}
opp\(_2\) = 227 = r\(^{114}\) and 114 \equiv 2 \pmod{4}
opp\(_3\) = 22 = r\(^{271}\) and 271 \equiv 3 \pmod{4}
opp\(_4\) = 185 = r\(^{121}\) and 121 \equiv 1 \pmod{4}

\(p = 457\)
\(\{r, x\} = \{13, 297\}\)
Game 1: \((1, 297, 64, 160)\)
\(x^m - 1 = 63 = r^{376}\)
opp\(_1\) = 296 = r\(^{453}\) and 453 \equiv 1 \pmod{4}
opp\(_2\) = 361 = r\(^{64}\) and 64 \equiv 0 \pmod{4}
opp\(_3\) = 298 = r\(^{47}\) and 47 \equiv 3 \pmod{4}
opp\(_4\) = 224 = r\(^{398}\) and 398 \equiv 2 \pmod{4}

3 Asymptotic Results

A fairly common methodology associated with establishing existence of designs is to utilize the following theorem, due to A. Weil, that relates to finite fields. The theorem and its proof can be found in [5].

**Theorem 3.1** Let \(q\) be a prime or a prime power and let \(\chi\) be a multiplicative character of \(GF(q)\) of order \(s > 1\). Let \(f \in GF(q)[x]\) be a monic polynomial of positive degree that is not an \(s\)-th power of a polynomial. Let \(w\) be the number of distinct roots of \(f\) in its splitting field over \(GF(q)\). Then for every \(a \in GF(q)\) we have \(\left| \sum_{x \in GF(q)} \chi(af(x)) \right| \leq (w - 1)\sqrt{q}\).
Theorem 3.2 Let $p$ denote a prime of the form $p = 2^kt+1$ where $k \geq 3$ and $t$ is odd. Let $r$ be a primitive root of $p$. The set of games $(1, x, x^m, -x) \otimes z^{di+2j}$, $0 \leq i \leq t-1, 0 \leq j \leq n-1$ forms the initial round of a $Z$-cyclic $SWh(p)$ if (1) $x$ is a non-square in $\mathbb{Z}_p$, (2) $x^2 - 1$ is a non-square in $\mathbb{Z}_p$, (3) $\prod_{i=1}^{k-2}(x^{2^i} + 1)$ is a non-square in $\mathbb{Z}_p$, (4) $x^3(x^{m-2} - x^{m-3} + \ldots - x + 1) \in C_0^4$ and (5) $x^3(x^{m-2} + x^{m-3} + \ldots + x + 1) \in C_0^4$.

Proof: Hypothesis (1) is identical to hypothesis (1) of Theorem 1.6. Hypotheses (2) and (3) imply that $x^m - 1 = (x^2 - 1)(\prod_{i=1}^{k-2}(x^{2^i} + 1)$ is a square in $\mathbb{Z}_p$. Thus hypothesis (2) of Theorem 1.6 is satisfied. Hypotheses (1) and (4) imply that $x^{m-2} - x^{m-3} + \ldots - x + 1$ is a non-square in $\mathbb{Z}_p$. Likewise hypotheses (1) and (5) imply that $x^{m-2} + x^{m-3} + \ldots + x + 1$ is a non-square in $\mathbb{Z}_p$. These latter facts combined with the observation that hypothesis (2) indicates that exactly one of $x + 1, x - 1$ is a square in $\mathbb{Z}_p$ enables one to conclude that the split opponent differences from the base game $(1, x, x^m, -x)$ can be partitioned as follows. The sets $G_1 = \{x + 1, x^m + x\}$ and $G_2 = \{x - 1, x^m - x\}$ are such that the elements in $G_i$ are either both squares in $\mathbb{Z}_p$ or both non-squares in $\mathbb{Z}_p$. Furthermore if the elements in $G_i$ are both squares then the elements in $G_j$ are both non-squares and vice-versa. Note that hypotheses (1) and (4) imply that $\frac{x^m + x}{x + 1} \in C_2^4$. Likewise hypotheses (1) and (5) imply that $\frac{x^m - x}{x - 1} \in C_2^4$. It now follows that hypothesis (3) of Theorem 1.6 is satisfied.

The sufficient conditions contained in Theorem 3.2 are such that, for a fixed $k \geq 3$, one can use Weil’s Theorem to obtain an asymptotic bound, say $N(k)$, beyond which these sufficient conditions are guaranteed to be satisfied. Let $\lambda_j$ denote a multiplicative character of order $j$, $j | (p - 1)$. Let $y \in \mathbb{Z}_p$. Define $\lambda_j(n) = e^{2\pi i b/j}$ if $y \in C_b^i$ and $\lambda_j(n) = 0$ if $y = 0$. For $z \in \mathbb{Z}_p$, let $f(z)$ denote a polynomial in $z$ that is not a $j$-th power of a polynomial. Set $I_{j,f} = 1 + \lambda_j(f(z)) + \lambda_j(f(z)^2) + \ldots + \lambda_j(f(z)^{j-1})$ and $O_{j,f} = (j - 1) - \lambda_j(f(z)) - \lambda_j(f(z)^2) - \ldots - \lambda_j(f(z)^{j-1})$. Observe that $I_{j,f} = j$ if $f(z) \in C_0^i$ and $I_{j,f} = 0$ if $f(z) \notin C_0^i$. Furthermore $O_{j,f} = j$ if $f(z) \notin C_0^i$ and $O_{j,f} = 0$ if $f(z) \in C_0^i$. Set $f_1(x) = x^2 - 1, f_2(x) = \prod_{i=1}^{k-2}(x^{2^i} + 1)$, $f_3(x) = x^3(x^{m-2} - x^{m-3} + \ldots - x + 1)$ and $f_4(x) = x^3(x^{m-2} + x^{m-3} + \ldots + x + 1)$. Consider the sum...
\[ S = \sum_{x \in \mathbb{Z}_p} O_{2,x} \cdot O_{2,f_1} \cdot O_{2,f_2} \cdot I_{4,f_3} \cdot I_{4,f_4} \]

If \( A \) denotes the set of all \( x \in \mathbb{Z}_p \) that satisfy the hypotheses of Theorem 3.2 then \( S = 128|A| \). Therefore \( A \) is not empty if \( S > 0 \). One proceeds by expanding \( S \) algebraically and considers every term in this expansion, other than the constant term, to be negative. Distribute, now, the summation over all these revised terms. In the general case if the constant term is \( a_0 \) then the summation of the constant term yields \( a_0p \). In the present case \( a_0 = 1 \). At this point one would want to apply Weil’s theorem. In order to do so it is required that all of the characters be of the same order. This can be accomplished via appropriate application(s) of Lemma 3.4 below. This being done, application of Weil’s Theorem to each remaining summation, results in \( S \geq p - \mu \sqrt{p} \) where \( \mu \) is a constant that depends on the sufficient conditions and, in turn, on \( k \). Thus \( S > 0 \) if \( p > \mu^2 \) and the asymptotic bound, \( N(k) \), equals \( \mu^2 \).

For the particular case \( k = 3 \), \( f_1(x) = x^2 - 1 \), \( f_2(x) = x^2 + 1 \), \( f_3 = x^3(x^2 - x + 1) \), \( f_4 = x^3(x^2 + x + 1) \). Consequently, \( \mu = 637 \) and \( N(3) = 405,769 \).

**Theorem 3.3** Splittable whist tournament designs exist for all primes of the form \( p = 8t + 1 \), \( t \) odd, except, possibly \( p = 41, 73, 89, 233, 281, 313, 409, 457, 601, 1433 \).

**Proof:** The materials above indicate that if \( p = 8t + 1 \), \( t \) odd, is a prime such that \( p \) is greater than the asymptotic bound \( N(3) = 405,769 \) then there must exist an \( x \in \mathbb{Z}_p \) for which the hypotheses of Theorem 3.2 are satisfied. For such \( p < N(3) \) a computer search has produced an appropriate \( x \) except for the possible exceptions listed in the statement of the theorem. The data is available from the authors.

For the exceptional cases see Section 4 below.

**Lemma 3.4** [3] Let \( q \) be a prime power. Suppose that \( s_1, s_2 \) are positive integers such that \( s_1|s_2|(q - 1) \). Then \( \chi_{s_1}(y) = \chi_{s_2}(y^{s_2/s_1}) \).
4 Liaw’s Construction

Several of the exceptional cases given in Theorem 1.8 and all but one of the exceptional cases listed in Theorem 3.3 can be removed by employing an alternative construction to the one given in Theorem 1.6. A useful alternative construction is one due to Y. S. Liaw [4]. Let $x$ be a non-square in $\mathbb{Z}_p$ and let $a$ be an integer such that $a \equiv (m - 1) \pmod{d}$. Consider the games $(1, x, -x, x^{a+1}) \otimes r^{di+2j}$, $0 \leq i \leq t - 1$, $0 \leq j \leq n - 1$. Appealing to Theorem 1.6, these games will form the initial round of a splittable whist tournament design if $(x+1)(x^a-1)$ is a square and the quadruple of integers $(a_1 = ind(x - 1), a_2 = ind(x(x^a + 1)), a_3 = ind(x^{a+1} - 1), a_4 = ind(2x))$ satisfies the hypotheses of Lemma 1.4.

**Theorem 4.1** Splittable whist tournament designs exist for all primes of the form $p = 8t + 1$, $t$ odd, except, possibly, for $p = 41$.

*Proof:* For the exceptional cases indicated in Theorem 3.3 solutions were obtained using Liaw’s Construction. The only failure occurred for $p = 41$. The data, in the format $\{p, r, x, a\}$, is as follows.

\[
\begin{align*}
\{73, 5, 59, 51\} & \quad \{89, 3, 3, 19\} & \quad \{233, 3, 3, 51\} \\
\{281, 3, 3, 123\} & \quad \{313, 10, 10, 203\} & \quad \{409, 21, 21, 59\} \\
\{457, 13, 13, 35\} & \quad \{601, 7, 7, 179\} & \quad \{1433, 3, 3, 3\}
\end{align*}
\]

Relative to Theorem 1.8 we have the following.

**Theorem 4.2** SWh$(p)$ exist for the following cases:

\begin{itemize}
  \item $k = 4$, $p = 113, 241, 337$;
  \item $k = 5$, $p = 97$;
  \item $k = 7$, $p = 641$.
\end{itemize}

*Proof:* Solutions for these cases can be obtained using Liaw’s Construction. The data, in the format $\{p, r, x, a\}$, is as follows.

\[
\begin{align*}
\{113, 3, 3, 39\} & \quad \{241, 7, 7, 231\} & \quad \{337, 10, 326, 23\} \\
\{97, 5, 38, 79\} & \quad \{641, 3, 202, 63\}
\end{align*}
\]

Several examples illustrating these latter results are given in Appendix II.
References


# Appendix I - Solution Data

The data is listed in the format \{p, r, x\} and is to be interpreted that, for \( p \) given, \( r \) and \( x \) are such that the hypotheses of Theorem 1.6 are satisfied.

(I.) \( k = 3 \)

\[
\begin{align*}
\{137,3,24\} & \quad \{409,21,21\} & \quad \{521,3,344\} & \quad \{569,3,337\} \\
\{617,3,243\} & \quad \{761,6,534\} & \quad \{809,3,580\} & \quad \{857,3,660\} \\
\{937,5,118\} & \quad \{953,3,749\} & \quad \{1033,5,973\} & \quad \{1049,3,404\} \\
\{1097,3,382\} & \quad \{1129,11,479\} & \quad \{1193,3,929\} & \quad \{1289,6,438\} \\
\{1321,13,804\} & \quad \{1433,3,94\} & \quad \{1481,3,404\} & \quad \{1609,7,1202\}
\end{align*}
\]

(II.) \( k = 4 \)

\[
\begin{align*}
\{401,3,3\} & \quad \{433,5,85\} & \quad \{593,3,429\} & \quad \{881,3,594\} \\
\{977,3,27\} & \quad \{1009,11,496\} & \quad \{1201,11,492\} & \quad \{1297,10,153\} \\
\{1361,3,3\} & \quad \{1489,14,14\} & \quad \{1553,3,27\} & \quad \{1777,5,1121\} \\
\{1873,10,274\} & \quad \{2129,3,1576\} & \quad \{2161,23,1658\} & \quad \{2417,3,347\} \\
\{2609,3,27\} & \quad \{2801,3,554\} & \quad \{2833,5,1604\} & \quad \{2897,3,922\}
\end{align*}
\]

(III.) \( k = 5 \)

\[
\begin{align*}
\{353,3,132\} & \quad \{673,5,393\} & \quad \{929,3,825\} & \quad \{1249,7,410\} \\
\{1697,3,435\} & \quad \{1889,3,567\} & \quad \{2017,5,125\} & \quad \{2081,3,1088\} \\
\{2273,3,2126\} & \quad \{2593,7,2475\} & \quad \{2657,3,1716\} & \quad \{3041,3,1469\} \\
\{3169,7,1857\} & \quad \{3361,22,565\} & \quad \{3617,3,486\} & \quad \{4001,3,644\} \\
\{4129,13,2856\} & \quad \{4513,7,3128\} & \quad \{5153,5,3423\} & \quad \{5281,7,3830\}
\end{align*}
\]

(IV.) \( k = 6 \)

\[
\begin{align*}
\{193,5,5\} & \quad \{449,3,241\} & \quad \{577,5,240\} & \quad \{1217,3,438\} \\
\{1601,3,1055\} & \quad \{2113,5,5\} & \quad \{2753,3,243\} & \quad \{3137,3,1315\} \\
\{4289,3,1298\} & \quad \{4673,3,4116\} & \quad \{4801,7,1361\} & \quad \{5441,3,606\} \\
\{5569,13,4004\} & \quad \{5953,7,4723\} & \quad \{6337,10,4676\} & \quad \{6977,3,3\} \\
\{7489,7,4938\} & \quad \{7873,5,6391\} & \quad \{8513,5,877\} & \quad \{8641,17,394\}
\end{align*}
\]
(V.) \( k = 7 \)
\[
\{1153,5,612\} \quad \{1409,3,27\} \quad \{2689,19,1979\} \quad \{3457,7,249\} \\
\{4481,3,388\} \quad \{4993,5,428\} \quad \{6529,7,1820\} \\
\{9601,13,6455\} \quad \{9857,5,428\} \quad \{6529,7,1820\} \\
\{12161,3,2610\} \quad \{13441,11,7164\} \quad \{15233,3,12842\} \\
\{16001,3,2187\} \quad \{18049,13,16856\} \quad \{19841,3,9596\}
\]

(VI.) \( k = 8 \)
\[
\{769,11,366\} \quad \{3329,3,2693\} \quad \{7937,3,5956\} \quad \{9473,3,2371\} \\
\{14081,3,2187\} \quad \{14593,5,12743\} \quad \{22273,5,17057\} \quad \{23297,3,27\} \\
\{26881,11,19330\} \quad \{30977,3,9121\} \quad \{31489,7,10126\} \quad \{36097,5,25789\} \\
\{37633,5,17494\} \quad \{40193,3,19683\} \quad \{41729,3,24770\} \quad \{43777,5,125\} \\
\{46337,3,27\}
\]

(VII.) \( k = 9 \)
\[
\{7681,17,5157\} \quad \{10753,11,10425\} \quad \{11777,3,9476\} \quad \{17921,3,15503\} \\
\{23041,11,10381\} \quad \{26113,7,9022\} \quad \{32257,15,11538\} \quad \{36353,3,31144\} \\
\{45569,3,32920\} \quad \{51713,3,41976\} \quad \{67073,3,21866\} \quad \{76289,3,44764\}
\]

Appendix II - Examples Illustrating Liaw’s Construction

\( k = 3 \)

\( p = 73, \ r = 5, \ x = 59, \ a = 51 \)

Game 1: \((1,59,14,71)\)

\( x = 59 = r^5 \)

\((x + 1)(x^a - 1) = 32 = r^{40} \)

\ opp1 = 58 = r^{43} \)

\ opp2 = 57 = r^{68} \)

\ opp3 = 45 = r^{13} \)

\ opp4 = 70 = r^{42} \)

opp indices (mod 4) : 3, 0, 1, 2
\( p = 89, r = 3, x = 3, a = 19 \)
Game 1: \((1, 3, 86, 73)\)
\[ x = 3 = r^1 \]
\[ (x + 1)(x^a - 1) = 34 = r^{22} \]
opp1 = 2 = \( r^{16} \)
opp2 = 76 = \( r^{67} \)
opp3 = 6 = \( r^{17} \)
opp4 = 72 = \( r^{50} \)
opp indices (mod 4) : 0, 3, 1, 2

\( k = 4 \)

\( p = 113, r = 3, x = 3, a = 39 \)
Game 1: \((1, 3, 110, 83)\)
\[ x = 3 = r^1 \]
\[ (x + 1)(x^a - 1) = 69 = r^{42} \]
opp1 = 2 = \( r^{12} \)
opp2 = 86 = \( r^{59} \)
opp3 = 6 = \( r^{13} \)
opp4 = 82 = \( r^{106} \)
opp indices (mod 4) : 0, 3, 1, 2

\( k = 5 \)

\( p = 97, r = 5, x = 38, a = 79 \)
Game 1: \((1, 38, 59, 62)\)
\[ x = 38 = r^{19} \]
\[ (x + 1)(x^a - 1) = 91 = r^{56} \]
opp1 = 37 = \( r^{91} \)
opp2 = 3 = \( r^{70} \)
opp3 = 76 = \( r^{53} \)
opp4 = 61 = \( r^{64} \)
opp indices (mod 4) : 3, 2, 1, 0