Domination in Bounded Interval Tolerance Graphs

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Abstract

In this paper we present efficient algorithms to solve domination set problem for the class of bounded tolerance graphs. The class of bounded tolerance graphs includes interval graphs and permtation graph as subclasses. Our solution to domination set problem has improved bounds for time complexity than the algorithm for the general co-comparability graphs which can as well be used for bounded tolerance graphs.

1. Introduction

Bounded interval tolerance graphs offer a more general model to represent wire-nets within a channel of a VLSI chip, than a interval graph model [14]. Interval tolerance graphs also provide a model to represent conflicts among events occurring within a block of time, when a tolerance of acceptable overlap is associated with each event. For this reason interest to examine problems in bounded interval tolerance graphs has grown [1]in recent past.

Interval tolerance graphs which were first introduced in [2] generalize both interval graphs and permutation graphs. Other characterizations of interval tolerance graphs place them within the class of trapezoid graphs [5], the class of alternatingly orientable graphs [4], the class of astroid-triple free graphs [2], and in the class of co-comparability graphs [2].

A set of vertices S in a graph $G = (V, E)$ is called a *dominating set* of G if every vertex in $(V \setminus S)$ is adjacent to some vertex in S. The *domination set problem* in a graph refers to the problem of finding the smallest set of vertices in the graph which form a dominating set. In general, the domination set problem is known to belong to the class of NP-complete problems [20], however, for several special classes of graphs [21], [23], [19], [22], [3] the problem has been solved in polynomial time.

Bounded tolerance graphs are co-comparability graphs [2] and correspond to partial orders which have interval dimension ≤ 2 [25], [4], therefore algorithm for the general co-comparability graphs [3] can as well be used to solve domination set problem in bounded tolerance graphs. Our solution, however, provides an improved complexity bound by taking advantage of the fact that tolerance graphs have a canonical square region embedding [4] in rectilinear plane. A pair of *interval tree* data structures [24] are used to represent the set of interval realizers of the bounded tolerance graph. The resulting algorithm has $\mathbf{O}(n^2)$ time complexity, which is an improvement over the $\mathbf{O}(n^3)$ time complexity of the algorithm for general co-comparability graphs.

Definitions appear in Section 2 and some characteristics are presented in Section 3 which provide the basis of the algorithm. The algorithm and analysis appear in Section 4.

2. Definitions

2.1. Interval Tolerance Graph

An undirected graph $G = (V, E)$ is called a *interval tolerance graph* if there exists a collection $\mathcal{I} = \{I_x \mid x \in V\}$ of closed intervals on a line and a set $\mathcal{T} = \{t_x \mid x \in V\}$ of positive numbers satisfying the condition $xy \in E \Leftrightarrow \mid I_x \cap I_y \mid \geq \min\{t_x, t_y\}$ where $\mid I \mid$ denotes the length of interval I. The pair $< \mathcal{I}, \mathcal{T} >$ is called *tolerance representation* of G. A tolerance representation $< \mathcal{I}, \mathcal{T} >$ is a *bounded* tolerance representation if $t_x \leq \mid I_x \mid$ for every $x \in V$ and a tolerance graph is called a *bounded inteval tolerance* graph if it admits a bounded tolerance representation.

Bounded interval tolerance representation for an example set of interval-tolerance pairs is shown in Figure 1 and the tolerance graph for the same example set appears in Figure 2.

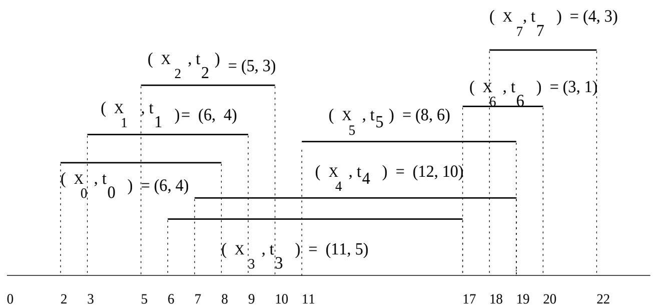

Figure 1. Tolerance representation for an example set of interval-tolerance pair

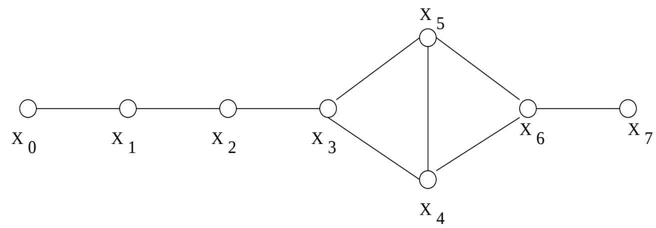

Figure 2. Tolerance Graph G for the example set of interval-tolerance pairs

2.2. Trepezoid Representation

Bounded interval tolerance graphs fall within the family of trepezoid graphs [13] where each vertex of the graph maps to a trepezoid and there is an edge between vertex x and vertex y in the graph whenever the intersection between the mappings of x and y is nonempty.

An interval-tolerance pair (I_x, t_x) has **T**repezoid **R**epresentation $TR(x)$ defined as follows: given two parallel lines L_1 and L_2, if $l(x)$ and $r(x)$ denote the left and right end points of interval I_x on the real line, then the trapezoid associated with vertex x is the convex hull of points $l(x)$, and $r(x) - t_x$ on line L_1 and the points $l(x) + t_x$, and r_x on line L_2.

For the example set of interval-tolerance pairs in Figure 1 the corresponding trepezoid representation appears in Figure 3.

2.3. Square Embedding

Bounded interval tolerance graphs entertain a canonical square embedding defined in [4]. Each vertex of the graph maps to a square embedding and there is an edge between vertex x and vertex y in the graph whenever the intersection between the shadows of the mappings of x and y on axis-1 or on axis-2 is nonempty.

An interval-tolerance pair (I_x, t_x) has **S**quare embedding **R**epresentation $SR(x)$ in two dimensional rectilinear plane defined as follows: if $l(x)$ and $r(x)$ denote the left and right end points of interval I_x on the real line, then the lower left corner of the square embedding is at the coordinates $(l(x), l(x)+t_x)$ and the upper right corner is at coordinates $(r(x) - t_x, r(x))$.

For the example set of interval-tolerance pairs in Figure 1 the corresponding square embeddings appears in Figure 4.

2.4. Interval Realizer

The pair of intervals $I_x^1 = [l(x), r(x) - t_x]$ and $I_x^2 = [l(x) + t_x, r(x)]$ are called *interval realizers* of I_x. In geometric terms, I_x^1 is mapping of I_x to an interval on axis-1 and I_x^2 is mapping of I_x to an interval on axis-2. Clearly $\mid I_x^1 \mid = \mid I_x^2 \mid$.

3 Characteristics

The complement graphs of bounded interval tolerance graphs correspond to partial order with interval dimension ≤ 2 [2]. That is, the partial order can be recovered from a family consisting of two or less interval orders. When the two interval orders are mapped to two parallel lines we obtain the trepezoid representation (definition 2.2) and when the two interval orders are mapped to two axes in the rectilinear plane we obtain the square embedding (definition 2.3). If G^1 is a graph represented by interval order $\mathcal{I}^1$ which has intervals of type I_x^1 (definition 2.4) and if G^2 is a graph represented by interval order $\mathcal{I}^2$ which has intervals of type I_x^2 (definition 2.4). Then the bounded interval tolerance graph is $G = G^1 \cup G^2$ and since G^1 and G^2

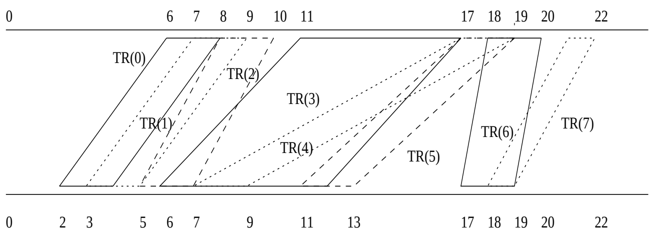

Figure 3. Trepezoid representation for the example set of interval-tolerance pairs

are co-comparability graphs of interval order P^1 and P^2 respectively, the comparability graph of $P^1 \cap P^2$ is the complement of G.

In this section we show that the geometrical relationship between placement of square embeddings corresponds to the partial order $P^1 \cap P^2$ of the complement of interval tolerance graphs. This property of bounded interval tolerance graphs allows us to work with geometrical representation instead.

If **l**ower **l**eft corner of a square embedding $SR(x)$ is denoted by $ll(x)$ and **u**pper **r**ight corner is denoted by $ur(x)$ the square embedding $SR(x)$ is said to be "less than" ($<$) square embedding $SR(y)$ if $ur(x) < ll(y)$ that is, $(r(x) - t_x < l(y)$ and $r(x) < l(y) + t_y$. In other words, the square embedding $SR(x)$ is physically to the "left and below" the square embedding $SR(y)$. When two square embeddings $SR(x)$ and $SR(y)$ are not related by $<$ order then they are said to be "incomparable". If $SR(x)$ and $SR(y)$ are incomparable, it will be denoted by $SR(x) \parallel SR(y)$. In this notation the relation between the square embeddings of Figure 4 are denoted as: $SR(0) \parallel SR(1)$, $SR(0) < SR(2)$, $SR(0) < SR(7)$ etc.

The following theorem establishes that if there is an edge (x, y) in the bounded tolerance graph i.e., $| I_x \cap I_y | \geq \min\{t_x, t_y\}$ then the square embeddings $SR(x)$ and $SR(y)$ are incomparable.

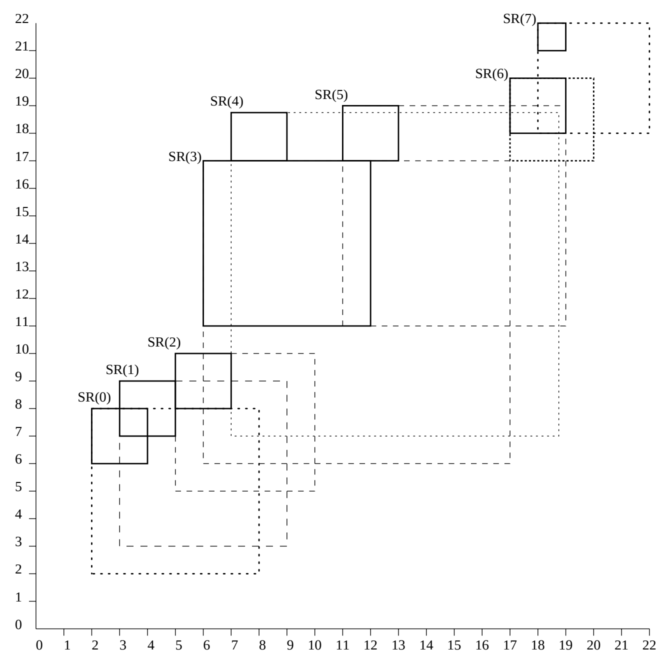

Figure 4. Square embeddings for the example set of interval-tolerance pairs

Theorem 3.1 *If* $| I_x \cap I_y | \geq \min\{t_x, t_y\}$ *for the interval-tolerance pairs* (I_x, t_x) *and* (I_y, t_y) *then the square* $SR(x)$ *and* $SR(y)$ *are incomparable.*

Proof:
Case 1. $(| I_x \cap I_y | \geq \min\{t_x, t_y\})$ and $| I_x \cap I_y | < I_x, I_y$
Without loss of generality, let $\min\{t_x, t_y\} = t_x$. Therefore $| I_x \cap I_y | \geq t_x$ and either **(a).** $| I_x \cap I_y | \geq t_y$ or **(b).**$| I_x \cap I_y | < t_y$
Square region embeddings for the two cases Case 1(a) and Case 1(b) are of the form illustrated in Figure 5. In either case $SR(x) \parallel SR(y)$. The alternate case when $\min\{t_x, t_y\} = t_y$ is analogous.

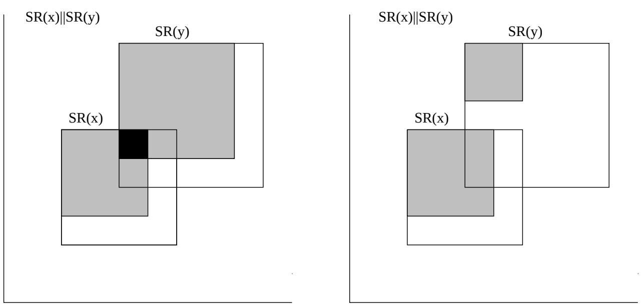

Figure 5. Case 1(a) and Case 1(b)

Case 2. $(| I_x \cap I_y | \geq \min\{t_x, t_y\})$ and $I_x \cap I_y \supseteq I_y$. Square embeddings for this case are of the form illustrated in Figure 6. In this case as well $SR(x) \parallel SR(y)$. The alternate case when $I_x \cap I_y \supseteq I_x$ is analogous.

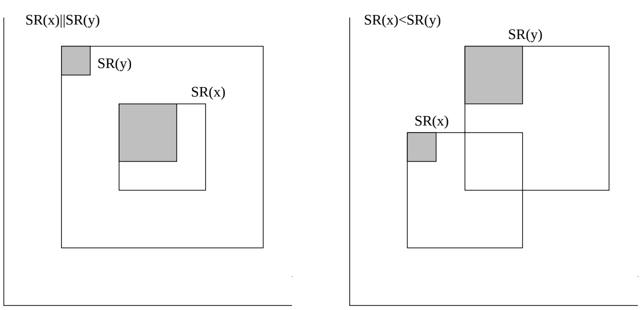

Figure 6. Case 2 and Case 3

Case 3. $(| I_x \cap I_y | < \min\{t_x, t_y\})$
Without loss of generality let $\min\{t_x, t_y\} = t_x$. Therefore, from the case condition, $| I_x \cap I_y | < t_x$ and also that $| I_x \cap I_y | < t_y$ The square embeddings for this case are of the form illustrated in Figure 6. In this case alone $SR(x) < SR(y)$.

Lemma 3.2 *There is an edge* (x, y) *in bounded interval tolerance graph* G *if and only if either* $I_x^1 \cap I_y^1 \neq \emptyset$ *or* $I_x^2 \cap I_y^2 \neq \emptyset$.

We say that there is an edge (x, y) in the graph whenever 'shadows' of square embeddings $SR(x)$ and $SR(y)$ overlap either on axis-1 or on axis-2.

4. Algorithms

The main idea behind the domination algorithm is based on construction of an auxiliary graph G^*. First, we introduce two new vertices s and t into the graph G that correspond to the least and the greatest element of the partial order $<$ (described in the Section 3). Second, directed edges are added from s to every vertex z in the graph G such that the element s in the partial order $(V \cup \{s, t\}, <)$ is the greatest lower bound of z. Similarly, directed edges are added from every vertex x in the graph G to t, such that the element t in the partial order $(V \cup \{s, t\}, <)$ is the lowest upper bound of x. Finally, a directed edge (u, v) is placed in the graph G whenever $SR(u) < SR(v)$ and that there is no element w such that $SR(u) < SR(w) < SR(v)$. The new directed graph is called the auxiliary graph G^*. The auxiliary graph G^* for the example interval tolerance representation of Figure 1 is shown in Figure 7. It is shown in Corollary 4.2 that the shortest directed path from s to t in G^* defines the minimum domination set for G.

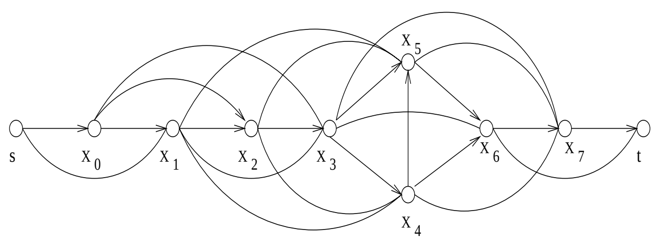

Figure 7. Auxiliary graph G^* for the example interval tolerance representation

Theorem 4.1 *Vertices on any path from vertex* s *to vertex* t*, in the auxiliary graph* G^* *for the tolerance graph* G *form a domination set in the graph* G

Sketch of the Proof: A set of vertices $\{x, y, z\}$ is an *asteroidal triple* if any two of them are connected by a path which avoids the neighborhood of the remaining vertex. Co-comparability graphs do not contain aseroidal triples, hence bounded tolerance graphs are asteroidal triple-free as well. The auxiliary graph G^*

retains the asteroidal triple-free property while adding new edges. Therefore, any path from s to t goes thru the neighborhood of every other vertex in G. Hence vertices on a path from s to t form a domination set.

Corollary 4.2 *Shortest path, from vertex s to vertex t, in the auxiliary graph G^* of a tolerance graph G passes thru the vertices that form the smallest domination set in the tolerance graph G.*

There are several alternate shortest paths from s to t in the auxiliary graph G^* for the example interval tolerance representation. For example, shortest paths $\{s \to x_1 \to x_5 \to x_7 \to t\}$ and $\{s \to x_0 \to x_2 \to x_6 \to t\}$ give minimum domination set $\{x_1, x_5, x_7\}$ and $\{x_0, x_2, x_6\}$ respectively.

The algorithm assumes that the tolerance interval graph is connected and its square embedding representation is available as input. A pair of *interval trees* [17], [24] $\mathcal{T}^1$ and $\mathcal{T}^2$, one each for interval realizer $\mathcal{I}^1$ and $\mathcal{I}^2$ are used as data structures. Each interval tree has $2n$ leaves for the corner points and has $O(\log n)$ height.

Algorithm Minimum_Domination (G);

Input: Tolerance Graph $G = (V, E)$;
Tolerance Representation $(\mathcal{I}, \mathcal{T})$
Output: Minimum Domination Set D
begin

1. Construct an auxiliary graph $G^* = (V^*, E^*)$ where:
   $V^* = V \cup \{s, t\}$ and
   $E^* = E \cup E_1 \cup E_2 \cup E_3$
   The edge sets E_1, E_2, and E_3 are defined by following rules:
   (a) $(u, v) \in E_1$ if there is no vertex w such that $SR(u) < SR(w) < SR(v)$.
   (b) $(s, z) \in E_2$ if there is no vertex h such that $SR(h) < SR(z)$
   (c) $(x, t) \in E_3$ if there is no vertex y such that $SR(y) > SR(x)$
2. Compute shortest path P^* from s to t in G^*.
3. Output the set of vertices on path P^* as the set D.

end.

Computation in step-(1)a. is carried out by finding "for an interval I_w, the set of intervals $LT_w^1 = \{I_u \mid I_u < I_w\}$, and the set of intervals $GT_w^1 = \{I_v \mid I_w < I_v\}$ in the interval realizer $\mathcal{I}^1$". Similarly the set LT_w^2, and the set GT_w^2 are found in the interval realizer $\mathcal{I}^2$. Intersection of the sets LT_w^1 and LT_w^2 gives the set of vertices u for which $SR(u) < SR(w)$. Similarly the intersection of the sets GT_w^1 and GT_w^2 gives the set of vertices v for which $SR(w) < SR(v)$ needed for step-1(a).

4.1. Algorithm Complexity

The two interval trees $\mathcal{T}^1$, and $\mathcal{T}^2$ are constructed for interval realizers $\mathcal{I}^1$ and $\mathcal{I}^2$ respectively. This construction requires $\mathbf{O}(n \log n)$ time. Computation needed for step-1(a) can be completed in $\mathbf{O}(\log n)$ time for each interval giving a $\mathbf{O}(n \log n)$ overall time for step-1. The shortest path computation in step-2 requires $\mathbf{O}(n^2)$and dominates the time complexity of the algorithm.

5. Conclusion

We have presented $\mathbf{O}(n^2)$ time algorithm to find domination set for the class of bounded tolerance graphs. This class includes both interval graphs and permutation graphs as subclasses. The algorithm takes advantage of the fact that bounded tolerance graphs have canonical square embedding in the rectilinear plane. This approach reduces the domination problem to a shortest path problem giving a much more efficient algorithm than the $\mathbf{O}(n^3)$ algorithm for the class of co-comparability graphs.

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